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URN: urn:nbn:de:gbv:ilm1-2018300019

Retrodigitalisierung der gleichnamigen Ausgabe:

Erschienen in:	Mathematical programming / A. - Berlin ; Heidelberg : Springer ; Amsterdam : North-Holland ; Amsterdam ; Lausanne [u.a.] : North-Holland, Elsevier Science. - Vol. 56 (1992), S. 91-119.
ISSN (print):	0025-5610
ISSN (online):	1436-4646
Auch online verfügbar:	DOI: 10.1007/BF01580896
Digitalisierung durch:	Universitätsbibliothek Ilmenau / ilmedia
Digitalisierungsjahr:	2018
Format:	TIFF, 300 DPI, 8 BPP

Mathematical Programming 56 (1992) 91–119
North-Holland

On stability in multiobjective programming – A stochastic approach

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Received 2 January 1989
Revised manuscript received 20 March 1990

In this paper we assume that a deterministic multiobjective programming problem is approximated by surrogate problems based on estimations for the objective functions and the constraints. Making use of a large deviations approach, we investigate the behaviour of the constraint sets, the sets of efficient points and the solution sets if the size of the underlying sample tends to infinity. The results are illustrated by applying them to stochastic programming with chance constraints, where (i) the distribution function of the random variable is estimated by the empirical distribution function, (ii) certain parameters have to be estimated.

Key words: Stability, stochastic approach, multiobjective programming, stochastic programming, chance constraints.

1. Introduction

Suppose that a decision maker is given a deterministic programming problem

$$\text{(P)} \qquad \min_{x\in\Gamma} \boldsymbol{f}(x),$$

where $\Gamma \subset \mathbb{R}^p$ is a nonempty closed set and $\boldsymbol{f}:\mathbb{R}^p \to \mathbb{R}^r$, and he is interested in the set of efficient points and the solution set (set of Pareto-optimal points) with respect to the usual partial ordering in $\mathbb{R}^r$ (which is generated by $\mathbb{R}^r_+$). However, in real-life situations he often does not completely know the objective functions or/and the constraint set Γ. He has to deal with estimators $\boldsymbol{f}_n$, Γ_n, where n in general denotes the size of the underlying sample. Furthermore, it may happen that solving the original problem requires too much effort and he is looking for surrogate problems (P_n) which are supposed to be easier to solve. Hence the question arises: Under what conditions concerning $\boldsymbol{f}$, Γ and the estimators can he expect that for instance the solution sets of the surrogate problems approximate the solution set of the original problem in a suitable manner if n tends to infinity? For problems (P) with one objective function only there are several approaches which help to give an answer. If the surrogate problems are deterministic one often can rely on the widely developed stability theory in parametric programming. A lot of papers devoted to stability in stochastic programming make use of these results too, for instance regarding the distribution function of the random variable as parameter [6, 13, 23, 25, 26, 27, 30, 32, 35].

Having in mind true stochastic surrogate problems one may ask for the asymptotic distribution of the solution to the approximate problems [4–7, 15, 16, 30–34, 37]. But aiming at such assertions at least certain Lipschitz properties are needed. Finally one may be tempted to apply stochastic convergence notions. The authors who use the epi-consistency approach derive conditions ensuring that the objective functions of the approximate problems epi-converge to the objective function of the original problem almost surely. This implies that cluster points of a sequence of solutions to the approximate problems are (with probability one) solutions to the true problem [7, 17]. In [14, 34] large deviations results are proved. They can be regarded as assertions on convergence in probability with additional convergence rate. (A more detailed discussion of stochastic stability concepts will be the topic of a forthcoming paper.)

With the present paper we continue the considerations in [34] and extend them to the multiobjective case. We use a unifying framework which covers important special cases. Papers devoted to stability for multiobjective stochastic problems are not known to us, but there are several results which are concerned with the deterministic case [20, 21, 29]. It turns out that the assertions given there remain true in the stochastic context (as far as they fit into our framework). Moreover, specializing the stochastic results proved in the following we may even obtain assertions which extend the existing literature in the deterministic case.

The paper is organized as follows: After introducing the model and proving necessary measurability statements we derive the basic results which describe the behaviour of the efficiency set and the solution set. Then the constraint set is dealt with in a rather general setting. Finally the results will be illustrated by specializing them to the following two cases:

(i) The objective functions are the expectations of functions of x and a random variable Z. Γ is given by chance constraints depending on Z. The distribution function of Z is estimated by the empirical distribution function.

(ii) The objective functions and Γ are as in case (i). However, we shall assume that the distribution function of Z is known up to certain parameters which have to be estimated.

With these investigations we also contribute to the theory of probabilistic constrained programs (cf. [12, 22, 36, 38]).

2. Preliminaries

Let the problem (P) be given. Assume that estimators $\boldsymbol{f}_n(x,\cdot)$, $n\in\mathbb{N}$, for $\boldsymbol{f}(x)$ are available, which are defined on the complete probability space $[\Omega, \mathcal{A}, P]$ and map into the measurable space $[\mathbb{R}^r, \mathcal{X}^r]$. $\mathcal{X}^r$ denotes the σ-field of Borel sets of $\mathbb{R}^r$. $\boldsymbol{f}_n$ is supposed to be $(\mathcal{X}^p\otimes\mathcal{A}, \mathcal{X}^r)$-measurable, sufficient conditions for this property are given in Section 6.2. Γ will be approximated by multifunctions Γ_n with measurable graphs. In our setting multifunctions with measurable graphs are measurable, i.e. $\Gamma_n^{-1}(A)=\{\omega\in\Omega: \Gamma_n(\omega)\cap A\neq\emptyset\}\in\mathscr{A}$ for every closed set $A\in\mathscr{L}^p$.

So we have the surrogate problems

$$(\mathrm{P}_n(\omega))\quad \min_{x\in\Gamma_n(\omega)} \boldsymbol{f}_n(x,\omega).$$

If we deal with a single component of $\boldsymbol{f}$ or $\boldsymbol{f}_n$ (or other vector-valued functions) we use the same letter without hold-face and add the corresponding index: f^j denotes the jth component of $\boldsymbol{f}$. Furthermore, we abbreviate

$$F:=\{\boldsymbol{f}(x): x\in\Gamma\}=\boldsymbol{f}(\Gamma),$$

$$F_n(\omega):=\{\boldsymbol{f}_n(x,\omega): x\in\Gamma_n(\omega)\}=\boldsymbol{f}_n(\Gamma_n(\omega),\omega).$$

The sets of efficient points (or efficiency sets) for the original problem (P) and the approximate problems $(\mathrm{P}_n(\omega))$ are explained by

$$E:=\{y\in F: \nexists\bar{y}\in F \text{ with } (\bar{y}\leqslant y\wedge\bar{y}\neq y)\},$$

$$E_n(\omega):=\{y\in F_n(\omega): \nexists\bar{y}\in F_n(\omega) \text{ with } (\bar{y}\leqslant y\wedge\bar{y}\neq y)\}.$$

$$X^E:=\{x\in\Gamma: \nexists\bar{x}\in\Gamma \text{ with } (\boldsymbol{f}(\bar{x})\leqslant\boldsymbol{f}(x)\wedge\boldsymbol{f}(\bar{x})\neq\boldsymbol{f}(x))\}$$

and

$$X_n^E(\omega):=\{x\in\Gamma_n(\omega): \nexists\bar{x}\in\Gamma_n(\omega) \text{ with } (\boldsymbol{f}_n(\bar{x},\omega)\leqslant\boldsymbol{f}_n(x,\omega)\wedge\boldsymbol{f}_n(\bar{x},\omega)\neq\boldsymbol{f}_n(x,\omega))\}$$

are the solution sets (or sets of Pareto-optimal points). Moreover, we introduce the sets of weakly efficient points

$$W:=\{y\in F: \nexists\bar{y}\in F \text{ with } \bar{y}<y\},$$

$$W_n(\omega):=\{y\in F_n(\omega): \nexists\bar{y}\in F_n(\omega) \text{ with } \bar{y}<y\}$$

and the corresponding (weak) solution sets

$$X^W:=\{x\in\Gamma: \nexists\bar{x}\in\Gamma \text{ with } \boldsymbol{f}(\bar{x})<\boldsymbol{f}(x)\},$$

$$X_n^W(\omega):=\{x\in\Gamma_n(\omega): \nexists\bar{x}\in\Gamma_n(\omega) \text{ with } \boldsymbol{f}_n(\bar{x},\omega)<\boldsymbol{f}_n(x,\omega)\},$$

where $(a^1,\ldots,a^r)^{\mathrm{T}}<(b^1,\ldots,b^r)^{\mathrm{T}}$, $a^i, b^i\in\mathbb{R}$, means that $a^i<b^i\ \forall i\in\{1,\ldots,r\}$.

Observe that by definition $E\subset W$ and $X^E\subset X^W$ for all problems under consideration.

Throughout the paper we shall deal with multifunctions having measurable graphs. Lemma 1 gives sufficient conditions for this property. Note that assumption (ii) enables us to include probabilities of certain events into the objectives (compare the treatment of chance constraints in Section 6).

Lemma 1. *Let one of the following conditions be satisfied*:

(i) *The functions $\boldsymbol{f}_n(\cdot,\omega)$ are continuous and the sets $\Gamma_n(\omega)$ are compact for almost all ω.*

(ii) *There exists a set $V_n\in\mathscr{L}^r$ with at most finitely many elements such that*

$$P\{\omega: \boldsymbol{f}_n(x,\omega)\in V_n\ \forall x\in\mathbb{R}^p\}=1.$$

Then the multifunctions F_n, E_n, W_n, X_n^E and X_n^W have measurable graphs.

Proof. We start by showing that Graph $F_n \in \mathscr{A}\otimes\mathscr{L}^r$ and F_n is closed-valued in both cases. Let n be fixed and let

$$\Omega_1 := \{\omega: f_n(\cdot,\omega) \text{ continuous and } \Gamma_n(\omega) \text{ compact}\},$$

$$\Omega_2 := \{\omega: f_n(x,\omega) \in V_n \ \forall x \in \mathbb{R}^p\}.$$

Suppose that (i) is satisfied. Then $F_n(\omega)$, $\omega \in \Omega_1$, is the image of a compact set via a continuous function, hence it is compact. By Corollary 1P and Theorem 1E in [24] we may conclude Graph $F_n \in \mathscr{A}\otimes\mathscr{L}^r$.

In case (ii) F_n is closed-valued by the finiteness of V_n. Therefore we shall show that $F_n^{-1}(V) \in \mathscr{A}$, where $V \subset V_n$, and employ Theorem 1E in [24].

Let $B_V(\omega) := \{x \in \mathbb{R}^p: f_n(x,\omega) \in V\}$. f_n being $(\mathscr{L}^p\otimes\mathscr{A}, \mathscr{L}^r)$-measurable, Graph $B_V \in \mathscr{A}\otimes\mathscr{L}^p$. Further

$$F_n^{-1}(V) = \{\omega: F_n(\omega) \cap V \neq \emptyset\} = \{\omega: \Gamma_n(\omega) \cap B_V(\omega) \neq \emptyset\} = \operatorname{dom}(\Gamma_n \cap B_V).$$

Since Graph $\Gamma_n \in \mathscr{A}\otimes\mathscr{L}^p$ and Graph $B_V \in \mathscr{A}\otimes\mathscr{L}^p$ we have Graph$(\Gamma_n \cap B_V) \in \mathscr{A}\otimes\mathscr{L}^p$ and finally $\operatorname{dom}(\Gamma_n \cap B_V) \in \mathscr{A}$.

In the next step we shall prove that measurability of $f_n(\cdot,\cdot)$, of Graph F_n and Graph Γ_n imply the measurability of X_n^E. In parts we use ideas of Papageorgiou [21].

From the definition of efficiency we have

$$X_n^E(\omega) = \{x \in \Gamma_n(\omega): (f_n(x,\omega) - \mathring{\mathbb{R}}_+^r) \cap F_n(\omega) = \emptyset\}$$

with $\mathring{\mathbb{R}}_+^r = \mathbb{R}_+^r \setminus \{0\}$. Hence

$$\text{Graph } X_n^E = \{(\omega, x): \omega \in \Omega, x \in \Gamma_n(\omega), (f_n(x,\omega) - \mathring{\mathbb{R}}_+^r) \cap (\omega) = \emptyset\}.$$

We introduce the multifunction $\tilde{\Phi}$ with

$$\tilde{\Phi}(\omega, x) := (f_n(x,\omega) - \mathring{\mathbb{R}}_+^r) \cap F_n(\omega)$$

and obtain

$$\text{Graph } X_n^E = \text{Graph } \Gamma_n \cap (\operatorname{dom} \tilde{\Phi})^c,$$

where $(\operatorname{dom}\tilde{\Phi})^c$ denotes the complement of the domain of $\tilde{\Phi}$. Furthermore, we investigate

$$\tilde{G}(\omega, x) := f_n(x,\omega) - \mathbb{R}_+^r.$$

Obviously

$$\tilde{\Phi}(\omega, x) = (\tilde{G}(\omega, x) \cap F_n(\omega)) \setminus (\{f_n(x,\omega)\} \cap F_n(\omega)).$$

$\tilde{G}$ and F_n being closed-valued and measurable, $\tilde{\Phi}$ is measurable by Theorem 4.5 in [10]. Thus $\operatorname{dom}\tilde{\Phi} \in \mathscr{A}\otimes\mathscr{L}^p$. Since Graph $\Gamma_n \in \mathscr{A}\otimes\mathscr{L}^p$, too, Graph $X_n^E \in \mathscr{A}\otimes\mathscr{L}^p$.

Finally, taking into account that

$$\text{Graph } E_n = \{(\omega, y): \omega \in \Omega, y \in F_n(\omega), (y - \mathring{\mathbb{R}}_+^r) \cap F_n(\omega) = \emptyset\},$$

we can proceed in the same way in order to show the measurability of Graph E_n.

If we consider X_n^W and W_n we replace $\mathring{\mathbb{R}}_+^r$ by $\operatorname{int}\mathbb{R}_+^r$ and $\tilde{\Phi}(\omega, x)$ by

$$\tilde{\Phi}_W(\omega, x) = (f_n(x, \omega) - \operatorname{int}\mathbb{R}_+^r) \cap F_n(\omega)$$
$$= (\tilde{G}(\omega, x) \cap F_n(\omega)) \setminus (\operatorname{bd}\tilde{G}(\omega, x) \cap F_n(\omega)).$$

bd $\tilde{G}$ is measurable due to Theorem 4.6 in [10]. □

Since we intend to derive assertions on the behaviour of sequences of multifunctions we need suitable convergence notions for multifunctions with measurable graphs and random functions. The definitions we present in the following have much in common with the definitions of convergence in probability used by Salinetti and Wets [28]. Thus our semiconvergence notions for multifunctions are nothing else but semi-versions of the convergence in probability completed with a convergence rate. The p-lower semiconvergence in probability of random functions, however, is not quite the same as the "lower" part in the definition of epi-convergence in probability in [28]. It is a specialized form which is adjusted to the fact that f is a deterministic function. The announced paper on stochastic stability concepts will explain these connections in detail.

Before we can give the definitions we still need some abbreviations. $U(M)$ denotes a neighbourhood of the set $M \subset \mathbb{R}^l$ (for a given l); $U_\varepsilon(M)$ is the ε-neighbourhood of M:

$$U_\varepsilon(M) := \{x \in \mathbb{R}^l : d(x, M) < \varepsilon\}, \quad \text{where } d(x, M) := \inf_{y \in M} \|x - y\|.$$

The radius of a neighbourhood $U\{x_0\}$ is defined by

$$\operatorname{rad} U\{x_0\} := \sup\{\varepsilon > 0 : U_\varepsilon\{x_0\} \subset U\{x_0\}\}.$$

Finally, a sequence $\xi = (\xi_n)_{n\in\mathbb{N}}$ will be called a convergence rate if $\xi_n \geqslant 0$ $\forall n \in \mathbb{N}$ and $\xi_n \to 0$. $\mathcal{K}^p$ is the family of compact sets $K \subset \mathbb{R}^p$.

Definition 1. A sequence $(G_n)_{n\in\mathbb{N}}$ of multifunctions $G_n : \Omega \to 2^{\mathbb{R}^p}$ with measurable graphs is said to be

(i) lower semiconvergent in probability to $G \subset \mathbb{R}^p$ with convergence rate ξ (abbreviated $G_n \xrightarrow{\text{l-prob}(\xi)} G$) if

$$\forall \varepsilon > 0, \quad \forall K \in \mathcal{K}^p, \quad P\{\omega : (G \setminus U_\varepsilon(G_n(\omega))) \cap K \neq \emptyset\} = o(\xi_n),$$

(ii) upper semiconvergent in probability to G with convergence rate ξ ($G_n \xrightarrow{\text{u-prob}(\xi)} G$) if

$$\forall \varepsilon > 0, \ \forall K \in \mathcal{K}^p, \quad P\{\omega : (G_n(\omega) \setminus U_\varepsilon(G)) \cap \neq \emptyset\} = o(\xi_n),$$

(iii) convergent in probability to G with convergence rate ($G_n \xrightarrow{\text{prob}(\xi)} G$) if

$$G_n \xrightarrow{\text{l-prob}(\xi)} G \wedge G_n \xrightarrow{\text{u-prob}(\xi)} G.$$

Definition 2. A sequence $(g_n)_{n\in\mathbb{N}}$ of functions $g_n:[\mathbb{R}^p\times\Omega, \mathscr{L}^p\otimes\mathscr{A}]\to[\mathbb{R}^1, \mathscr{L}^1]$ is said to be

(i) p-lower semicontinuously convergent in probability to $g:\mathbb{R}^p\to\mathbb{R}^1$ on a set $M\subset\mathbb{R}^p$ with convergence rate ξ (abbreviated $g_n \xrightarrow[M]{\text{pl-prob}(\xi)} g$) if

$$\forall\varepsilon>0,\ \forall x_0\in M,\ \exists U^\varepsilon\{x_0\}\in\mathscr{K}^p,$$

$$P\left\{\omega:\inf_{x\in U^\varepsilon\{x_0\}} g_n(x,\omega)-g(x_0)\leqslant-\varepsilon\right\}=\mathrm{o}(\xi_n),$$

(ii) p-upper semicontinuously convergent in probability to g on M with convergence rate ξ ($g_n \xrightarrow[M]{\text{pu-prob}(\xi)} g$) if

$$-g_n \xrightarrow[M]{\text{pl-prob}(\xi)} -g.$$

(iii) p-continuously convergent in probability to g on M with convergence rate ξ ($g_n \xrightarrow[M]{\text{p-prob}(\xi)} g$) if

$$g_n \xrightarrow[M]{\text{pl-prob}(\xi)} g \ \wedge\ g_n \xrightarrow[M]{\text{pu-prob}(\xi)} g.$$

The letter p is to indicate that we use a pointwise condition. Sufficient conditions for $g_n \xrightarrow[M]{\text{pl-prob}(\xi)} g$ will be given in Section 6.

The measurability of Graph G_n, $n\in\mathbb{N}$, implies that all events occurring in Definition 1 are elements of $\mathscr{A}$. The $(\mathscr{L}^p\otimes\mathscr{A}, \mathscr{L}^1)$-measurability of the random function g_n plays a corresponding role for Definition 2, compare Lemma III.39 in [2].

One cannot expect that p-semicontinuous convergence in probability of $(g_n)_{n\in\mathbb{N}}$ to g guarantees semicontinuity of g. But we have the following assertion:

Proposition 1. *If* $g_n \xrightarrow[\mathbb{R}^p]{\text{p-prob}(\xi)} g$, *then* g *is continuous.*

Proof. Assume that there are an $x_0\in\mathbb{R}^p$, an $\alpha>0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n\in\mathbb{R}^p$, $x_n\to x_0$ and $|g(x_n)-g(x_0)|>2\alpha$ $\forall n\in\mathbb{N}$. The definition of p-continuous convergence in probability assigns to α and each x_n, $n\in\mathbb{N}\cup\{0\}$, a neighbourhood $U^\alpha\{x_n\}$ with

$$P\left\{\omega:\sup_{x\in U^\alpha\{x_n\}}|g_n(x,\omega)-g(x_n)|\geqslant\alpha\right\}=\mathrm{o}(\xi_n).$$

Clearly $x_k\in U^\alpha\{x_0\}$ $\forall k\geqslant k_0$. Hence, because of

$$\begin{aligned}2\alpha<|g(x_k)-g(x_0)|&\leqslant|g_n(x_k,\omega)-g(x_k)|+|g_n(x_k,\omega)-g(x_0)|\\&\leqslant\sup_{x\in U^\alpha\{x_k\}}|g_n(x,\omega)-g(x_k)|+\sup_{x\in U^\alpha\{x_0\}}|g_n(x,\omega)-g(x_0)|\end{aligned}$$

for $k \geqslant k_0$, $\forall n \in \mathbb{N}$, $\forall \omega \in \Omega$, the equation

$$P\left\{\omega \colon \sup_{x \in U^{\alpha}\{x_0\}} |g_n(x,\omega) - g(x_0)| \geqslant \alpha \vee \sup_{x \in U^{\alpha}\{x_k\}} |g_n(x,\omega) - g(x_k)| \geqslant \alpha\right\} = 1$$

holds. This contradicts the p-continuous convergence in probability. □

In general we have to deal with a family of sequences $\{(g_n^j)_{n\in\mathbb{N}}, j \in \{1, \ldots, k\}\}$, p-(semi)continuously converging in probability on M. Then we use the neighbourhoods $U^{\varepsilon}\{x\} := \bigcap_{j\in\{1,\ldots,k\}} U_j^{\varepsilon}\{x\}$, where $U_j^{\varepsilon}\{x\}$ is given by the definition of p-(semi)continuous convergence in probability of $(g_n^j)_{n\in\mathbb{N}}$ to g^j on M. The family $\{U^{\varepsilon}\{x\}, x \in M\}$, will be called the family of neighbourhoods associated to $\{(g_n^j)_{n\in\mathbb{N}}, j \in \{1, \ldots, k\}\}$ and ε.

3. Stability of the efficiency set

We start by investigating the behaviour of $(F_n)_{n\in\mathbb{N}}$. Consider the following assumptions:

(VC1) $\Gamma_n \xrightarrow{\text{l-prob}(\xi)} \Gamma$;

(VC2) $\Gamma_n \xrightarrow{\text{u-prob}(\xi)} \Gamma$;

(VC3) there exists a set $C \in \mathcal{K}^p$ such that

$$P\{\omega \colon \Gamma_n(\omega) \not\subset C\} = o(\xi_n);$$

(VF) $f_n^j \xrightarrow[\mathbb{R}^p]{\text{p-prob}(\xi)} f^j$, $j \in \{1, \ldots, r\}$.

Note that the presupposed closedness of Γ, (VC1) and (VC3) imply the compactness of Γ.

Lemma 2. (i) *If* (VC1) *and* (VF) *are satisfied, then* $F_n \xrightarrow{\text{l-prob}(\xi)} F$.

(ii) *If* (VC2), (VC3) *and* (VF) *are satisfied, then* $F_n \xrightarrow{\text{u-prob}(\xi)} F$.

Proof. (i) Let $\varepsilon > 0$, $K \in \mathcal{K}^p$ and $n \in N$ be given.

We consider a finite cover $\{U_{\varepsilon/2}\{y_1\}, \ldots, U_{\varepsilon/2}\{y_k\}\}$ of the compact set cl $F \cap K$. f being continuous, there is a finite family of open sets $\{\tilde{U}\{x_l\} = f^{-1}(U_{\varepsilon/2}\{y_l\}) \cap U_{\varepsilon/2}(\Gamma)$, $l = 1, \ldots, k$, $x_l \in \Gamma$, $f(x_l) = y_l\}$, which forms a cover of $f^{-1}(\text{cl } F \cap K)\Gamma$. Furthermore, the definition of p-continuous convergence in probability assigns to $\tilde{\varepsilon} = \varepsilon/(2\sqrt{r})$ and each $x_l \in \Gamma$ a neighbourhood $U^{\tilde{\varepsilon}}\{x_l\}$. Let

$$U\{x_l\} := U^{\tilde{\varepsilon}}\{x_l\} \cap \tilde{U}\{x_l\}, \qquad \delta := \min_{l\in\{1,\ldots,k\}} \text{rad } U\{x_l\}$$

and

$$\hat{K} := \text{cl} \bigcup_{l\in\{1,\ldots,k\}} U\{x_l\},$$

Now assume that $(F \setminus U_\varepsilon(F_n(\omega))) \cap K \neq \emptyset$. Then we find a $y_0(\omega) \in F \cap K$ which does not belong to $U_\varepsilon(F_n(\omega))$. To $y_0(\omega)$ there exists an $x_0(\omega) \in \boldsymbol{f}^{-1}(F \cap K) \cap \Gamma$ with $y_0(\omega) = \boldsymbol{f}(x_0(\omega))$. Clearly there is an x_l such that $x_0(\omega) \in \tilde{U}\{x_l\}$. If $U\{x_l\} \cap \Gamma_n(\omega) = \emptyset$, we obtain $(\Gamma \setminus U_\delta(\Gamma_n(\omega))) \cap \hat{K} \neq \emptyset$. Otherwise, assume that there is an $x_n(\omega) \in U\{x_l\} \cap \Gamma_n(\omega)$. Then

$$\|\boldsymbol{f}_n(x_n(\omega), \omega) - \boldsymbol{f}(x_l)\| \geqslant \|\boldsymbol{f}_n(x_n(\omega), \omega) - \boldsymbol{f}(x_0(\omega))\| - \|\boldsymbol{f}(\chi_0(\omega)) - \boldsymbol{f}(x_l)\| \geqslant \tfrac{1}{2}\varepsilon.$$

Consequently there is a j with

$$\sup_{x \in U^{\tilde{\varepsilon}}\{x_l\}} |f_n^j(x, \omega) - f^j(x_l)| \geqslant \tilde{\varepsilon}.$$

Summarizing,

$$\begin{aligned} &P\{\omega\colon (F \setminus U_\varepsilon(F_n(\omega))) \cap K \neq \emptyset\} \\ &\quad \leqslant P\{\omega\colon (\Gamma \setminus U_\delta(\Gamma_n(\omega))) \cap \hat{K} \neq \emptyset\} \\ &\qquad + \sum_{l=1}^{k} \sum_{j=1}^{r} P\left\{\omega\colon \sup_{x \in U^{\tilde{\varepsilon}}\{x_l\}} |f_n^j(x, \omega) - f^j(x_l)| \geqslant \tilde{\varepsilon}\right\} = \mathrm{o}(\xi_n). \end{aligned}$$

(ii) Again, let $\varepsilon > 0$, a compact set K and $n \in \mathbb{N}$ be given, and consider the system of neighbourhoods $\{U^{\tilde{\varepsilon}}\{x\}, x \in C\}$, associated to $\{(f_n^j)_{n \in \mathbb{N}}, j \in \{1, \ldots, r\}\}$, and $\tilde{\varepsilon} = \varepsilon/(2\sqrt{r})$. Choose δ in such a way that $(\|x - x_0\| < \delta \wedge x, x_0 \in C)$ implies $|f^j(x) - f^j(x_0)| < \tilde{\varepsilon}\ \forall j \in \{1, \ldots, r\}$ and define $U\{x\} := U_{\delta/2}\{x\} \cap \operatorname{int} U^{\tilde{\varepsilon}}\{x\}$. Then there is a finite cover $\{U\{x_l\}, x_l \in C, l = 1, \ldots, k\}$, of the compact set C.

Now let ω be such that there exists a $y_n(\omega) \in F_n(\omega) \cap K$ with $y_n(\omega) \notin U_\varepsilon(F)$. To $y_n(\omega)$ we find an $x_n(\omega) \in \Gamma_n(\omega)$, $y_n = f_n(x_n(\omega), \omega)$. If $x_n(\omega)$ does not belong to C, we obtain $\Gamma_n(\omega) \not\subset C$ and can exploit (VC3). Otherwise $x_n(\omega) \in U\{x_l\}$ for some l. Suppose that $U\{x_l\} \cap U_{\delta/2}(\Gamma) = \emptyset$. Then we have $(\Gamma_n(\omega) \setminus U_{\delta/2}(\Gamma)) \cap C \neq \emptyset$.

Finally, let $x_n(\omega) \in U\{x_l\}$ and $\tilde{x}_n(\omega) \in U_\delta\{x_l\} \cap \Gamma$. In this case

$$\|\boldsymbol{f}_n(x_n(\omega), \omega) - \boldsymbol{f}(x_l)\| \geqslant \|\boldsymbol{f}_n(x_n(\omega), \omega) - \boldsymbol{f}(\tilde{x}_n(\omega))\| - \|\boldsymbol{f}(\tilde{x}_n(\omega)) - \boldsymbol{f}(x_l)\|$$

$\geqslant \tfrac{1}{2}\varepsilon$.

Thus

$$\begin{aligned} &P\{\omega\colon (F_n(\omega) \setminus U_\varepsilon(F)) \cap K \neq \emptyset\} \\ &\quad \leqslant P\{\omega\colon (\Gamma_n(\omega) \not\subset C\} + P\{\omega\colon (\Gamma_n(\omega) \setminus U_{\delta/2}(\Gamma)) \cap C \neq \emptyset\} \\ &\qquad + \sum_{l=1}^{k} \sum_{j=1}^{r} P\left\{\omega\colon \sup_{x \in U^{\tilde{\varepsilon}}\{x_l\}} |f_n^j(x, \omega) - f^j(x_l)| \geqslant \tilde{\varepsilon}\right\} = \mathrm{o}(\xi_n). \qquad \square \end{aligned}$$

Lemma 2 is the stochastic variant of Lemma 4.4.1 in [29] and Proposition 4 in [20].

A desirable property of the sequence of surrogate problems $(\mathrm{P}_n)_{n \in \mathbb{N}}$ would be the upper semiconvergence in probability of $(E_n)_{n \in \mathbb{N}}$ to E. But as known from the deterministic case, in general one can only expect that the efficiency sets of (P_n) tend to belong to the set of weakly efficient points of (P) (cf. [29]). Concerning the lower semiconvergence in probability of $(E_n)_{n \in \mathbb{N}}$ to E the situation is better: It holds under rather weak conditions.

Theorem 1. *Let the following assumptions be satisfied:*

(A1) *F is compact;*

(A2) $F_n \xrightarrow{\text{prob}(\xi)} F$;

(A3) *there exists a set $\tilde{C} \in \mathcal{K}^r$ with $P\{\omega\colon F_n(\omega) \not\subset \tilde{C}\} = o(\xi_n)$.*

Then:

$$\text{(i)} \qquad [P\{\omega\colon E_n(\omega) \neq \emptyset\} = o(\xi_n)] \Rightarrow [E_n \xrightarrow{\text{l-prob}(\xi)} E],$$

$$\text{(ii)} \qquad W_n \xrightarrow{\text{u-prob}(\xi)} W.$$

Before we prove this theorem we shall give a lemma.

Lemma 3. *Let F be closed. Then*

$$\forall K \in \mathcal{K}^r,\ \forall C \in \mathcal{K}^r,\ \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall y_0 \in E \cap K,$$

$$\exists y_1 \in E \text{ with } U_\delta\{y_1\} \subset U_\varepsilon\{y_0\},$$

$$[(U_\delta\{y_1\} + \mathbb{R}^r_-) \setminus U_\varepsilon\{y_0\}] \cap U_\delta(F) \cap C = \emptyset.$$

Proof. Suppose to the contrary that there are compact sets K, C, an $\varepsilon > 0$ and sequences $(\delta_n)_{n\in\mathbb{N}}$, $\delta_n \downarrow 0$, and $(y_{0n})_{n\in\mathbb{N}}$, $y_{0n} \in E \cap K$, with the property that for all $y_{1n} \in E$ with $U_{\delta_n}\{y_{1n}\} \subset U_\varepsilon\{y_{0n}\}$ an element $z_n(y_{1n})$ of $[(U_{\delta_n}\{y_{1n}\} + \mathbb{R}^r_-) \setminus U_\varepsilon\{y_{0n}\}] \cap U_{\delta_n}(F) \cap C$ exists.

W.l.o.g. we may assume that $y_{0n} \to \hat{y}_0 \in \text{cl } E$. We consider the set $U_{\varepsilon/2}\{\hat{y}_0\} \cap E$. By definition of $\hat{y}_0$ it is nonempty.

Now, fix an element $\hat{y}_1 \in U_{\varepsilon/2}\{\hat{y}_0\} \cap E$. Clearly $U_{\delta_n}\{\hat{y}_1\} \subset U_\varepsilon\{y_{0n}\}$ $\forall n \geq n_0$. According to the assumption there is a sequence $(z_n)_{n\in\mathbb{N}} = (z_n(\hat{y}_1))_{n\in\mathbb{N}}$ with the properties $z^j_n \leq \hat{y}^j_1 + \delta_n$, $z_n \notin U_{\varepsilon/2}\{\hat{y}_0\}$ $\forall v \geq n_1$ and $z_n \in U_{\delta_n}(F) \cap C$. $(z_n)_{n\in\mathbb{N}}$ contains a converging subsequence: $z_{nk} \to z_0 \in C$. Consequently $z^j_0 \leq \hat{y}^j_1$, $z_0 \neq \hat{y}_1$ and $z_0 \in F$. This implies $\hat{y}_1 \notin E$. Thus $U_{\varepsilon/2}\{\hat{y}_0\} \cap E = \emptyset$ in contradiction to $\hat{y}_0 = \lim_{n\to\infty} y_{0n}$, $y_{0n} \in E$. $\square$

Proof of Theorem 1. (i) Let $\varepsilon > 0$, $K \in \mathcal{K}^r$ and $n \in \mathbb{N}$ be fixed and let ω be such that $(E \setminus U_\varepsilon(E_n(\omega))) \cap K \neq \emptyset$. Then there is a $y_0(\omega) \in E \cap K$ with $y_0(\omega) \notin U_\varepsilon(E_n(\omega))$. The preceding lemma assigns to ε, K, $\tilde{C}$ and $y_0(\omega)$ a ball $U_\delta\{y_1(\omega)\}$ with $y_1(\omega) \in E$. Note that δ does not depend on ω.

Suppose that $U_\delta\{y_1(\omega)\} \cap F_n(\omega) = \emptyset$. Hence $(F \setminus U_\delta(F_n(\omega))) \cap F \neq \emptyset$ and we exploit (A2).

In the following we assume that there is a $y_n(\omega) \in U_\delta\{y_1(\omega)\} \cap F_n(\omega)$. Since $y_n(\omega) \notin E_n(\omega)$ we either have $E_n(\omega) = \emptyset$ or we find $\tilde{y}_n(\omega) \in E_n(\omega)$ with $\tilde{y}_n(\omega) \in (U_\delta\{y_1(\omega)\} + \mathbb{R}^r_-) \setminus U_\varepsilon\{y_0(\omega)\}$. According to Lemma 3, $[(U_\delta\{y_1(\omega)\} + \mathbb{R}^r_-) \setminus U_\varepsilon\{y_0(\omega)\}] \cap U_\delta(F) \cap \tilde{C} = \emptyset$. If $\tilde{y}_n(\omega) \notin \tilde{C}$ we make use of (A3). Otherwise $\tilde{y}_n(\omega) \notin U_\delta(F)$, and we obtain $(F_n(\omega) \setminus U_\delta(F)) \cap \tilde{C} \neq \emptyset$. It remains to apply (A2).

(ii) Let $\varepsilon>0$, $K\in\mathscr{K}^r$ be fixed and define

$$C_\varepsilon := \{y\in F: d(y, W)\geqslant\varepsilon\}.$$

To each $\tilde{y}\in C_\varepsilon$ there exists a $y_0\in W$ with $y_0<\tilde{y}$. Hence there are an $\alpha(\tilde{y}, y_0)>0$ and open balls $\tilde{U}\{\tilde{y}\}$ and $\tilde{U}_0\{y_0\}$ with

$$\hat{y}_0+\alpha(\tilde{y}, y_0)\mathbf{1}<\hat{y}\quad \forall\hat{y}_0\in\tilde{U}_0\{y_0\},\ \forall\hat{y}\in\tilde{U}\{\tilde{y}\}\quad (\mathbf{1}=(1,\dots,1)^{\mathrm{T}}\in\mathbb{R}^r).$$

The sets $\{\tilde{U}\{\tilde{y}\}, \tilde{y}\in C_\varepsilon\}$ being an open cover of the compact set C_ε, we can select a finite cover $\{\tilde{U}\{y_l\}, l=1,\dots,k\}$. By y_{0l} we denote the corresponding elements of W. Let

$$\alpha := \min_{l\in\{1,\dots,k\}} \alpha(y_l, y_{0l}),\qquad V := \bigcup_{l\in\{1,\dots,k\}} \tilde{U}\{y_l\},$$

$$\delta_2 := \min_{l\in\{1,\dots,k\}} \operatorname{rad} \tilde{U}_0\{y_{0l}\}.$$

In the next step we show that we find a $\delta_1>0$ such that $U_{\delta_1}(F)\subset V\cup U_\varepsilon(W)=:\hat{M}$.

Assume that there are sequences $(\delta_{1n})_{n\in\mathbb{N}}$, $\delta_{1n}\downarrow 0$, and $(z_n)_{n\in\mathbb{N}}$ with $z_n\in U_{\delta_{1n}}(F)$, $z_n\notin\hat{M}$. W.l.o.g. $(z_n)_{n\in\mathbb{N}}$ is converging: $z_n\to z_0\in F$. Since $F\subset\hat{M}$ and $\hat{M}$ is open we obtain $z_n\in\hat{M}$ $\forall n\geqslant n_0$ in contradiction to the assumption.

Now, let $n\in\mathbb{N}$ and suppose that $(W_n(\omega)\setminus U_\varepsilon(W))\cap\neq\emptyset$. Then there is an $y_n(\omega)\in W_n(\omega)$, $y_n(\omega)\notin U_\varepsilon(W)$. $y_n(\omega)\notin V$ implies $(F_n(\omega)\setminus U_{\delta_1}(F))\cap\tilde{C}\neq\emptyset$ or $F_n(\omega)\not\subset\tilde{C}$. If $y_n(\omega)\in V$ there is a y_l with $y_n(\omega)\in\tilde{U}\{y_l\}$. According to the definition of $\tilde{U}\{y_l\}$ and $\tilde{U}_0\{y_{0l}\}$ we have $\hat{y}_0+\alpha\mathbf{1}<y_n(\omega)$ $\forall\hat{y}_0\in\tilde{U}_0\{y_{0l}\}$. Since $y_n(\omega)\in W_n(\omega)$ the intersection $F_n(\omega)\cap\tilde{U}_0\{y_{0l}\}$ must be empty. Therefore $(F\setminus U_{\delta_2}(F_n(\omega)))\cap F\neq\emptyset$. □

Corollary. *Let the assumptions* (VC1)–(VC3) *and* (VF) *be satisfied. Then the assertions of Theorem* 1 *hold.*

Proof. (VF) implies the continuity of $\boldsymbol{f}$, hence F is compact. Because of Lemma 2 the condition (A2) is fulfilled. It remains to verify (A3).

Let $\varepsilon>0$ be fixed and choose a finite open cover $\{U\{x_1\},\dots,U\{x_k\}\}$ of C, where $U\{x_l\}\subset U^{\tilde{\varepsilon}}\{x_l\}$ and $\|\boldsymbol{f}(x)-\boldsymbol{f}(x_l)\|\leqslant\frac{1}{2}\varepsilon$ $\forall x\in U\{x_l\}$. $\{U^{\varepsilon}\{x_l\}, x_l\in C\}$ is the system of neighbourhoods associated to $[(f_n^j)_{n\in\mathbb{N}}, j\in\{1,\dots,r\}]$, and $\tilde{\varepsilon}=\varepsilon/(2\sqrt{r})$. Take $\tilde{C}:=\operatorname{cl} U_\varepsilon(\boldsymbol{f}(\bigcup_{l\in\{1,\dots,k\}}U\{x_l\}))$ and suppose that $F_n(\omega)\not\subset\tilde{C}$. Hence there is an $x_n(\omega)\in\Gamma_n(\omega)$ with $\boldsymbol{f}_n(x_n(\omega),\omega)\not\subset\tilde{C}$. If $x_n(\omega)\notin C$, we make use of (VC3).

Otherwise $x_n(\omega)\in U\{x_l\}$ for some l. Consequently

$$\|\boldsymbol{f}_n(x_n(\omega),\omega)-\boldsymbol{f}(x_l)\|\geqslant\|\boldsymbol{f}_n(x_n(\omega),\omega)-\boldsymbol{f}(x_n(\omega))\|-\|\boldsymbol{f}(x_n(\omega))-\boldsymbol{f}(x_l)\|$$
$$\geqslant\tfrac{1}{2}\varepsilon,$$

and we apply (VF). □

Naturally the question arises under what conditions $W=E$ is fulfilled.

Proposition 2. *Let one of the following conditions be satisfied*:

(VE1) *Γ is convex and the functions f^j, $j\in\{1,\dots,r\}$, are strictly explicitly quasiconvex, i.e.*

$$[f^j(x_1)\leqslant f^j(x_2), x_1\neq x_2]\ \Rightarrow\ [f^j(\lambda x_1+(1-\lambda)x_2)<f^j(x_2)\ \forall\lambda\in(0,1)].$$

(VE2) *Γ is strictly convex, the functions f^j, $j \in \{1, \ldots, r\}$, are convex and have the following property*:

For all $x \in \Gamma$ there exists a $d \in \mathbb{R}^p$ such that for $j \in \{1, \ldots, r\}$ either $f^j_+(x, d) < 0$ or $f^j(\tilde{x}) > f^j(x)$ $\forall \tilde{x} \in U\{x\} \setminus \{x\}$ holds. ($f^j_+(x, d)$ denotes the directional derivative of f^j at x in direction d.)

(VE3) *For all y_λ with $y_\lambda = \lambda y_0 + (1-\lambda) y_1$, $y_0 \in F$, $y_1 \in F$, $y_0 \neq y_1$, $\lambda \in (0, 1)$, the set $(y_\lambda + \operatorname{int} \mathbb{R}^r_-) \cap F$ is nonempty.*

Then $W = E$.

The first assertion may be found in several papers. Moreover, its proof is similar to the first part of the proof to the second assertion, therefore we omit it.

Proof. Assume that there is an $y_0 \in W$ with $y_0 \notin E$. Then there exists an $y_1 \in \boldsymbol{f}(\Gamma)$ such that $y_1^j \leq y_0^j$ $\forall j \in \{1, \ldots, r\}$ and $y_1^{j_0} < y_0^{j_0}$ for at least one j_0. To y_0 and y_1 we find $x_0 \in \Gamma$ and $x_1 \in \Gamma$ with $y_0 = \boldsymbol{f}(x_0)$ and $y_1 = \boldsymbol{f}(x_1)$ and consider $x_\lambda := \lambda x_0 + (1-\lambda) x_1$, $\lambda \in (0, 1)$ arbitrary. If there is a $j_1 \in \{1, \ldots, r\}$ such that $f^{j_1}(x_\lambda) < f^{j_1}(\tilde{x})$ $\forall \tilde{x} \in U\{x_\lambda\} \setminus \{x_\lambda\}$, convexity implies $f^{j_1}(x_\lambda) < f^{j_1}(x_0)$. Consequently there is a neighbourhood $\hat{U}\{x_\lambda\}$ of x_λ with $f^{j_1}(x) < f^{j_1}(x_0)$ $\forall x \in \hat{U}\{x_\lambda\}$. Otherwise one finds a direction d and an $\alpha_0 > 0$ such that $x_\lambda + \alpha d \in \Gamma$ and $f^j(x_\lambda + \alpha d) < f^j(x_\lambda) \leq f^j(x_0)$, $0 < \alpha \leq \alpha_0$. Hence $y_0 \notin W$ in contradiction to the assumption.

In order to prove the third assertion one only has to take into account that $y_\lambda \leq \max\{y_0, y_1\}$ and $y < y_\lambda$ for all $y \in (y_\lambda + \operatorname{int} \mathbb{R}^r_-) \cap F$ holds. $\square$

(VE3) is a slight generalization of the strict convexity of F which is used in [21].

4. Stability of the solution set

In [29] stability results for the solution set are derived from the assertions for the efficiency set. We prefer to prove the results for the solution sets directly. Thus it will turn out that the convergence assumptions for $(f^j_n)_{n \in \mathbb{N}}$ can be weakened.

Theorem 2. *In addition to* (VC1)–(VC3) *let the following assumptions be satisfied*:

(VF1) $f^j_n \xrightarrow[\Gamma \setminus X^W]{\text{pl-prob}(\xi)} f^j$, *$f^j$ l.s.c. on Γ, $j \in \{1, \ldots, r\}$*;

(VF2) $f^j_n \xrightarrow[X^W]{\text{pu-prob}(\xi)} f^j$, *$f^j$ u.s.c. on X^W, $j \in \{1, \ldots, r\}$.*

Then $X^W_n \xrightarrow{\text{u-prob}(\xi)} X^W$.

Proof. Partly the proof corresponds to the proof of Theorem 1(ii). In order to emphasize the similarities we shall use the same notations as far as possible.

Let $\varepsilon>0$, $K\in\mathcal{K}^p$ be fixed and define

$$C_\varepsilon := \{x\in\Gamma\colon d(x, X^W)\geq\varepsilon\}.$$

To each $\tilde{x}\in C_\varepsilon$ we associate an $x_0 = x_0(\tilde{x})\in X^W$ with $\boldsymbol{f}(x_0)<\boldsymbol{f}(\tilde{x})$. (The existence of such an x_0 is ensured by our assumptions.) Further, because of the semicontinuity of f^j we find an $\alpha(\tilde{x})>0$ and open balls $\tilde{U}\{\tilde{x}\}$ and $\tilde{U}_0\{x_0\}$ with $\boldsymbol{f}(\hat{x}_0)+2\alpha(\tilde{x})\mathbf{1}<\boldsymbol{f}(\hat{x})$ $\forall\hat{x}_0\in\tilde{U}_0\{x_0\}, \forall\hat{x}\in\tilde{U}\{\tilde{x}\}$. The definition of p-lower semicontinuous convergence in probability of $\{(f_n^j)_{n\in\mathbb{N}}, j\in\{1,\ldots,r\}\}$ assigns to each $\tilde{x}\in C_\varepsilon$ and $\alpha(\tilde{x})$ a neighbourhood $U^{\alpha(\tilde{x})}\{\tilde{x}\}$. Let $U\{\tilde{x}\} := \tilde{U}\{\tilde{x}\}\cap\operatorname{int} U^{\alpha(\tilde{x})}\{\tilde{x}\}$, $\tilde{x}\in C_\varepsilon\cap K$. The sets $U\{\tilde{x}\}$, $\tilde{x}\in C_\varepsilon$, being an open cover of C_ε, we can select a finite cover $\{U\{x_l\}, l=1,\ldots,k\}$. By x_{0l} we denote the elements of X^W corresponding to x_l and by $U^{\alpha(x_l)}\{x_{0l}\}$ the neighbourhoods associated to x_{0l} and $\alpha(x_l)$ via the definition of p-upper semicontinuous convergence in probability of $\{(f_n^j)_{n\in\mathbb{N}}, j\in\{1,\ldots,r\}\}$. Thus with $\alpha_l := \alpha(x_l)$ and $U_0\{x_{0l}\} := \tilde{U}_0\{x_{0l}\}\cap\operatorname{int} U^{\alpha_l}\{x_{0l}\}$ we have

$$\boldsymbol{f}(\hat{x}_0)+2\alpha_l\mathbf{1}<\boldsymbol{f}(\hat{x})\quad\forall\hat{x}_0\in U_0\{x_{0l}\},\ \forall\hat{x}\in U\{x_l\}.$$

Further we shall use the following notations:

$$V := \bigcup_{l\in\{1,\ldots,k\}} U\{x_l\},\qquad \delta_2 := \min_{l\in\{1,\ldots,k\}} \operatorname{rad} U_0\{x_{0l}\}.$$

As in the proof of Theorem 1(ii) we can show that a $\delta_1>0$ with $U_{\delta_1}(\Gamma)\subset V\cup U_\varepsilon(X^W)$ exists.

Now, let $n\in\mathbb{N}$ and suppose that $(X_n^W(\omega)\setminus U_\varepsilon(X^W))\cap K\neq\emptyset$. Then there is an $x_n(\omega)\in X_n^W(\omega)$ with $x_n(\omega)\notin U_\varepsilon(X^W)$. If $x_n(\omega)\notin V$, we obtain $(\Gamma_n(\omega)\setminus U_{\delta_1}(\Gamma))\cap C\neq\emptyset$ or $\Gamma_n(\omega)\not\subset C$. Otherwise $x_n(\omega)$ belongs to some $U\{x_l\}$, $l\in\{1,\ldots,k\}$. If we have $U_0\{x_{0l}\}\cap\Gamma_n(\omega)=\emptyset$, then $(\Gamma\setminus U_{\delta_2}(\Gamma_n(\omega)))\cap\Gamma\neq\emptyset$. Finally, assume that there is an $\hat{x}_n(\omega)\in U_0\{x_{0l}\}\cap\Gamma_n(\omega)$. Since $x_n(\omega)\in X_n^W(\omega)$ we find a $j_0\in\{1,\ldots,r\}$ with $f_n^{j_0}(x_n(\omega),\omega)\leq f_n^{j_0}(\hat{x}_n(\omega),(\omega)$. Hence

$$f_n^{j_0}(x_n(\omega),\omega)-f^{j_0}(x_l)\leq -\tfrac{1}{2}[f^{j_0}(x_l)-f^{j_0}(x_{0l})]$$

or

$$f_n^{j_0}(\hat{x}_n(\omega),\omega)-f^{j_0}(x_{0l})\geq\tfrac{1}{2}[f^{j_0}(x_l)-f^{j_0}(x_{0l})].$$

Therefore

$$\inf_{x\in U^{\alpha_l}\{x_l\}} f_n^{j_0}(x,\omega)-f^{j_0}(x_l)\leq\alpha_l$$

or

$$\sup_{x\in U^{\alpha_l}\{x_{0l}\}} f_n^{j_0}(x,\omega)-f^{j_0}(x_{0l})\geq\alpha_l.$$

Summarizing,

$$\begin{aligned}
&P\{\omega: (X_n^W(\omega)\setminus U_\varepsilon(X^W))\cap K\neq\emptyset\}\\
&\quad\leqslant P\{\omega: (\Gamma_n(\omega)\setminus U_{\delta_1}(\Gamma))\cap C\neq\emptyset\}+P\{\omega: \Gamma_n(\omega)\not\subset C\}\\
&\qquad+P\{\omega: (\Gamma\setminus U_{\delta_2}(\Gamma_n(\omega)))\cap\Gamma\neq\emptyset\}\\
&\qquad+\sum_{l=1}^{k}\sum_{j=1}^{r}P\left\{\omega: \inf_{x\in U^{\alpha_l}\{x_l\}} f_n^j(x,\omega)-f^j(x_l)\leqslant-\alpha_l\right\}\\
&\qquad+\sum_{l=1}^{k}\sum_{j=1}^{r}P\left\{\omega: \sup_{x\in U^{\alpha_l}\{x_l\}} f_n^j(x,\omega)-f^j(x_{0l})\geqslant\alpha_l\right\}=\mathrm{o}(\xi_n). \qquad \square
\end{aligned}$$

In the next theorem the sets

$$C_\varepsilon(x_0):=\{x\in\Gamma: d(x,x_0)\geqslant\varepsilon\},\quad x_0\in X^E,\ \varepsilon>0,$$

will play an important role.

Theorem 3. *In addition to* (VC1)–(VC3) *and* (VF2) *let the following assumptions be satisfied*:

(VF1′) $f_n^j \xrightarrow[\Gamma]{\text{pl-prob}(\xi)} f^j$, f^j *l.s.c. on* Γ, $j\in\{1,\dots,r\}$;

(VF3) $\forall\varepsilon>0,\ \exists\delta>0,\ \forall x_0\in\operatorname{cl}X^E,\ \exists x_1\in X^E$ *with* $U_\delta\{x_1\}\subset U_\varepsilon\{x_0\}$,

$\forall x\in C_\varepsilon(x_0),\ \exists j_0\in\{1,\dots,r\},\ f^{j_0}(x)>f^{j_0}(x_1)$;

(VX) $P\{\omega: X_n^E(\omega)=\emptyset\}=\mathrm{o}(\xi_n)$.

Then $X_n^E \xrightarrow{\text{l-prob}(\xi)} X^E$.

Proof. Let $\varepsilon>0$ be given and choose a finite cover $\{U_{\varepsilon/2}\{x_{0l}\},\ x_{0l}\in\operatorname{cl}X^E, l=1,\dots,k\}$ of the compact set $\operatorname{cl}X^E$. To $\frac{1}{2}\varepsilon$ and each x_{0l} we fix $\delta>0$ and an $x_{1l}\in X^E$ such that $U_\delta\{x_{1l}\}\subset U_{\varepsilon/2}\{x_{0l}\}$ and

$$\forall x\in C_{\varepsilon/2}(x_{0l}),\ \exists j_0(x_{1l},x),\quad f^{j_0(x_{1l},x)}(x)>f^{j_0(x_{1l},x)}(x_{1l}).$$

Let l, x_{1l} and $x\in C_{\varepsilon/2}(x_{0l})$ be fixed. Because of the semicontinuity of f^j there are an $\alpha(x_{1l},x)>0$ and open neighbourhoods $\tilde U_l\{x\}$ and $\tilde U_{0x}\{x_{1l}\}$ such that

$$f^{j_0(x_{1l},x)}(\hat x)>f^{j_0(x_{1l},x)}(\hat x_1)+2\alpha(x_{1l},x),\quad \forall\hat x\in\tilde U_l\{x\},\ \forall\hat x_1\in\tilde U_{0x}\{x_{1l}\}.$$

Note that the letter "l" at $U_l\{x\}$ is to indicate the dependence on x_{1l} and not the radius of the neighbourhood. Further, consider

$$U_l\{x\}:=\tilde U_l\{x\}\cap\operatorname{int}U^{\alpha(x_{1l},x)}\{x\},$$

where $U^{\alpha(x_{1l},x)}\{x\}$ is the neighbourhood associated to $\{(f_n^j)_{n\in\mathbb{N}}, j\in\{1,\ldots,r\}\}$, $\alpha(x_{1l}, x)$ and x by the definition of p-lower semicontinuous convergence in probability. The family $\{U_l\{x\}, x\in C_{\varepsilon/2}(x_{0l})\}$ being an open cover of the compact set $C_{\varepsilon/2}(x_{0l})$, we can select a finite cover $\{U_l\{x_{lm}\}, m=1,\ldots,k_l\}$.

Now, let

$$\tilde{U}_0\{x_{1l}\} := \bigcup_{m\in\{1,\ldots,k_l\}} \tilde{U}_{0x_{lm}}\{x_{1l}\},$$

$$\alpha_{lm} := \alpha(x_{1l}, x_{lm}), \qquad \alpha_l := \min_{m\in\{1,\ldots,k_l\}} \alpha_{lm}$$

and

$$U_0\{x_{1l}\} := \tilde{U}_0\{x_{1l}\} \cap \operatorname{int} U^{\alpha_l}\{x_{1l}\} \cap U_\delta\{x_{1l}\},$$

where $U^{\alpha_l}\{x_{1l}\}$ is determined by the p-upper semicontinuous convergence in probability of $\{(f_n^j)_{n\in\mathbb{N}}, j\in\{1,\ldots,r\}\}$. Thus we have families $\{U_0\{x_{1l}\}, l=1,\ldots,k\}$ and $\{U_l\{x_{lm}\}, l=1,\ldots,k;\ m=1,\ldots,k_l\}$ such that (with $j_{lm} := j_0(x_{1l}, x_{lm})$)

$$f^{j_{lm}}(\hat{x}) > f^{j_{lm}}(\hat{x}_1) + 2\alpha_{lm}, \quad \forall \hat{x}\in U_l\{x_{lm}\},\ \forall \hat{x}_1 \in U_0\{x_{1l}\}.$$

Finally we introduce

$$V_l := \bigcup_{m\in\{1,\ldots,k_l\}} U_l\{x_{lm}\}, \qquad \delta_2 := \min_{l\in\{1,\ldots,k\}} \operatorname{rad} U_0\{x_{1l}\}.$$

As in the proof of Theorem 1(ii) we show the existence of a $\delta_1>0$ with $U_{\delta_1}(\Gamma)\subset \bigcap_{l\in\{1,\ldots,k\}} (V_l \cup U_{\varepsilon/2}\{x_{0l}\})$.

Now let $K\in\mathcal{K}^p$ and $n\in\mathbb{N}$ be given and suppose that $(X^E\setminus U_\varepsilon(X_n^E(\omega))\cap K\neq\emptyset$. Hence there is an $x_0(\omega)\in X^E\cap K$ with $x_0(\omega)\notin U_\varepsilon(X_n^E(\omega))$. Obviously we find an $x_{0l}\in \operatorname{cl} X^E$ such that $x_0(\omega)\in \hat{U}_{\varepsilon/2}\{x_{0l}\}$. According to the construction of δ, x_{1l} and $U_0\{x_{1l}\}$ we have $U_{\delta_2}\{x_{1l}\}\subset U_\delta\{x_{1l}\}\subset U_{\varepsilon/2}\{x_{0l}\}\subset U_\varepsilon\{x_0(\omega)\}$. If $\Gamma_n(\omega)\cap U_{\delta_2}\{x_{1l}\}=\emptyset$, we obtain $(\Gamma\setminus U_{\delta_2}(\Gamma_n(\omega)))\cap\Gamma\neq\emptyset$. Otherwise there is an $x_n(\omega)\in\Gamma_n(\omega)\cap U_{\delta_2}\{x_{1l}\}$. $x_n(\omega)$ does not belong to $X_n^E(\omega)$, hence $X_n^E(\omega)=\emptyset$ or we find an $\tilde{x}_n(\omega)\in X_n^E(\omega)$ with

$$f_n^j(\tilde{x}_n(\omega),\omega)\leq f_n^j(x_n(\omega),\omega) \quad \forall j\in\{1,\ldots,r\}. \tag{$*$}$$

Furthermore, since $\tilde{x}_n(\omega)\notin U_\varepsilon\{x_0(\omega)\}$, we may conclude $\tilde{x}_n(\omega)\notin U_{\varepsilon/2}\{x_{0l}\}$, hence $\tilde{x}_n(\omega)\in V_l$, or $d(\tilde{x}_n(\omega),\Gamma)\geq\delta_1$.

In the second case $(\Gamma_n(\omega)\setminus U_{\delta_1}(\Gamma))\cap C\neq\emptyset$ or $\Gamma_n(\omega)\not\subset C$ follows.

Finally, let $\tilde{x}_n(\omega)\in V_l$. Clearly there is an m such that $\tilde{x}_n(\omega)\in U_l\{x_{lm}\}$. Then we obtain from $(*)$ for all $j\in\{1,\ldots,r\}$,

$$f_n^j(\tilde{x}_n(\omega),\omega) - f^j(x_{lm}) \leq -\tfrac{1}{2}[f^j(x_{lm}) - f^j(x_{1l})]$$

or

$$f_n^j(x_n(\omega),\omega) - f^j(x_{1l}) \geq \tfrac{1}{2}[f^j(x_{lm}) - f^j(x_{1l})].$$

Taking into account that

$$f^{j_{lm}}(x_{lm}) - f^{j_{lm}}(x_{1l}) > 2\alpha_{lm},$$

we obtain

$$\inf_{x\in U^{\alpha_{lm}}\{x_{lm}\}} f_n^{j_{lm}}(x,\omega)-f^{j_{lm}}(x_{lm})\leq -\alpha_{lm}$$

or

$$\sup_{x\in U^{\alpha_{lm}}\{x_{1l}\}} f_n^{j_{lm}}(x,\omega)-f^{j_{lm}}(x_{1l})\geq \alpha_{lm}.$$

Summarizing,

$$\begin{aligned}
&P\{\omega\colon (X^E\setminus U_\varepsilon(X_n^E(\omega)))\cap K\neq\emptyset\}\\
&\quad\leq P\{\omega\colon (\Gamma\setminus U_{\delta_2}(\Gamma_n(\omega)))\cap\neq\emptyset\}+P\{\omega\colon X_n^E(\omega)=\emptyset\}\\
&\qquad+P\{\omega\colon (\Gamma_n(\omega)\setminus U_{\delta_1}(\Gamma))\cap C\neq\emptyset\}+P\{\omega\colon \Gamma_n(\omega)\not\subset C\}\\
&\qquad+\sum_{l=1}^{k}\sum_{m=1}^{k_l}\sum_{j=1}^{r}\left[P\left\{\omega\colon \inf_{x\in U^{\alpha_{lm}}\{x_{lm}\}} f_n^j(x,\omega)-f^j(x_{lm})\leq-\alpha_{lm}\right\}\right.\\
&\qquad\qquad\left.+P\left\{\omega\colon \sup_{x\in U^{\alpha_{lm}}\{x_{1l}\}} f_n^j(x,\omega)-f^j(x_{1l})\geq\alpha_{lm}\right\}\right]=\mathrm{o}(\xi_n).\qquad\square
\end{aligned}$$

Sufficient conditions for (VF3) are given in the following proposition.

Proposition 3. *Let the functions $f^j, j\in\{1,\ldots,r\}$, be l.s.c. on Γ and Γ be compact. Furthermore, suppose that one of the conditions* (VE1), (VE2), (VE3) *is satisfied or that $\boldsymbol{f}$ is one-to-one. Then* (VF3) *is fulfilled.*

Proof. Assume to the contrary

$$\exists\varepsilon>0,\ \forall\delta>0,\ \exists x_0\in\operatorname{cl}X^E,\ \forall x_1\in X^E \text{ with } U_\delta\{x_1\}\subset U_\varepsilon\{x_0\},$$
$$\exists x\in C_\varepsilon(x_0),\ \forall j\in\{1,\ldots,r\},\quad f^j(x)\leq f^j(x_1).$$

Choose $(\delta_n)_{n\in\mathbb{N}}$, $\delta_n\downarrow 0$, and a corresponding sequence $(x_{0n})_{n\in\mathbb{N}}$, $x_{0n}\in\operatorname{cl}X^E$. W.l.o.g. we can suppose that $(x_{0n})_{n\in\mathbb{N}}$ converges: $x_{0n}\to\hat{x}_0\in\operatorname{cl}X^E$. We fix an element $\hat{x}_1\in U_{\varepsilon/2}\{\hat{x}_0\}\cap X^E$. Clearly $U_{\delta_n}\{\hat{x}_1\}\subset U_\varepsilon\{x_{0n}\}\ \forall n\geq n_0$. According to the assumption there is a sequence $(x_n)_{n\in\mathbb{N}}=(x_n(\hat{x}_1))_{n\in\mathbb{N}}$ with $x_n\in C_\varepsilon(x_{0n})$ and $f^j(x_n)\leq f^j(\hat{x}_1)$ $\forall j\in\{1,\ldots,r\}$. f^j being l.s.c. on Γ, we obtain $f^j(\tilde{x})\leq f^j(\hat{x}_1)\ \forall j\in\{1,\ldots,r\}$ for each cluster point $\tilde{x}$ of the sequence $(x_n)_{n\in\mathbb{N}}$. The existence of cluster points $\tilde{x}\in C_{\varepsilon/2}(\hat{x}_0)$ is guaranteed by the compactness of Γ and $U_{3\varepsilon/4}\{\hat{x}_0\}\cap C_\varepsilon(x_{0n})=\emptyset\ \forall n\geq n_1$. If $\boldsymbol{f}$ is one-to-one, we may conclude $f^{j_0}(\tilde{x})<f^{j_0}(\hat{x}_1)$ for at least one $j_0\in\{1,\ldots,r\}$ in contradiction to $\hat{x}_1\in X^E$.

(VE1) or (VE2) being fulfilled, one can construct an $x_\lambda=\lambda\hat{x}_1+(1-\lambda)\tilde{x}$, $\lambda\in(0,1)$, with $f^j(x_\lambda)<f^j(\hat{x}_1)\ \forall j\in\{1,\ldots,r\}$ or an $\tilde{x}_\lambda=x_\lambda+\alpha d\in\Gamma$ with this property (compare the proof of Proposition 2).

Under condition (VE3) to $y_\lambda=\lambda\boldsymbol{f}(\hat{x}_1)+(1-\lambda)\boldsymbol{f}(\tilde{x})$, $\lambda\in(0,1)$, there is a $\tilde{y}\in(y_\lambda+\operatorname{int}\mathbb{R}_-^r)\cap\boldsymbol{f}(\Gamma)$. Consequently, for $\hat{x}\in\Gamma$ with $\boldsymbol{f}(\hat{x})=\tilde{y}$ the relation $f^j(\hat{x})<f^j(\hat{x}_1)\ \forall j\in\{1,\ldots,r\}$ follows. $\square$

It is easy to see that even weaker variants of (VE1)–(VE3) imply the assertion of Proposition 3. Thus, provided that the functions f^j are convex, it is for instance sufficient that only one f^{j_0} is strictly explicitly quasiconvex.

Note, however, that there are examples where $W=E$ holds but (VF3) is not satisfied.

Summarizing we can conclude that the conditions (VC1)–(VC3), (VX), (VF1′) and (VF2) and additionally one of the assumptions (VE1), (VE2), (VE3) guarantee the convergence in probability (with given convergence rate) of $(X_n^E)_{n\in\mathbb{N}}$ to X^E.

If we cannot expect that one of the conditions (VE1)–(VE3) holds we still have the assertion of Theorem 2 and we can prove that the random sets X_n^E tend to touch the set X^E.

Theorem 4. *Let the assumptions of Theorem* 2 *and* (VX) *be satisfied. Then*

$$\forall \varepsilon>0,\quad P\left\{\omega\colon \inf_{x\in \operatorname{cl} X_n^E(\omega)} d(x, X^E)\geqslant\varepsilon\right\}=o(\xi_n).$$

Proof. The measurability of $\inf_{x\in\operatorname{cl} X_n^E} d(x, X^E)$ is ensured by Theorem 2K in [24].

To prove the assertion of Theorem 4 we make use of the "one-dimensional" surrogate problem

$$(\mathrm{P^S})\qquad \min_{x\in\Gamma}\frac{1}{r}\sum_{j=1}^{r} f^j(x)$$

with solution set X and the associated approximate problems

$$(\mathrm{P}_n^{\mathrm S}(\omega))\quad \min_{x\in\Gamma_n(\omega)}\frac{1}{r}\sum_{j=1}^{r} f^j(x,\omega)$$

with solution sets $X_n(\omega)$.

Because of the implication

$$\left[\inf_{x\in U^{\varepsilon}\{x_0\}}\frac{1}{r}\sum_{j=1}^{r}(f_n^j(x,\omega)-f^j(x_0))\leqslant-\varepsilon\right]$$

$$\Rightarrow\left[\exists j_0\in\{1,\ldots,r\}\text{ with }\inf_{x\in U^{\varepsilon}\{x_0\}} f_n^{j_0}(x,\omega)-f^{j_0}(x_0)\leqslant-\varepsilon\right],$$

(VF1) is valid for the objective function $\tilde f(x):=(1/r)\sum_{j=1}^{r} f^j(x)$.

In the same way we show that (VF2) is fulfilled. Then by Theorem 2,

$$\forall\varepsilon>0,\quad P\{\omega\colon (X_n(\omega)\setminus U_\varepsilon(X))\cap C\neq\emptyset\}=o(\xi_n).$$

Now, let $\varepsilon>0$ and $n\in\mathbb{N}$ be fixed and suppose ω to be such that $\emptyset\neq X_n^E(\omega)\subset C$ and $d(x,X^E)\geqslant\varepsilon\ \forall x\in\operatorname{cl} X_n^E(\omega)$. Since $X_n(\omega)\subset X_n^E(\omega)$ and $X\subset X^E$, we obtain $d(x,\tilde x)\geqslant\varepsilon\ \forall\tilde x\in X,\ \forall x\in X_n(\omega)$, hence $(X_n(\omega)\setminus U_\varepsilon(X))\cap C\neq\emptyset$. □

The large deviations approach presented in this paper enables us to derive assertions on the convergence of the sequences under consideration in the "almost-surely" sense, too. To give an example we derive a result on the upper semiconvergence of X_n^E. Let $\varepsilon>0$, $C\in\mathcal{K}^p$ be given and define

$$A_\varepsilon^n:=\{\omega\colon (X_n^E(\omega)\setminus U_\varepsilon(X^E))\cap C\neq\emptyset\},$$
$$B^n:=\{\omega\colon \Gamma_n(\omega)\not\subset C\}.$$

Proposition 4. *Assume that for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(A_\varepsilon^n) < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} P(B^n) < \infty$$

holds. Then the cluster points of the sequence $(x_n(\omega))_{n\in\mathbb{N}}$, $x_n(\omega) \in X_n^E(\omega)$, *are elements of* $\operatorname{cl} X^E$ *for almost all* ω.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ with $x_n(\omega) \in X_n^E(\omega)$ $\forall \omega \in \Omega$ be given and suppose that there are an $\hat{\varepsilon} > 0$ and a set $\Omega_0 \in \Omega$ such that $P(\Omega_0) > 0$ and for all $\omega \in \Omega_0$ there exists a converging subsequence $(x_{n_k}(\omega))_{k\in\mathbb{N}}$ of $(x_n(\omega))_{n\in\mathbb{N}}$ with $\lim_{k\to\infty} x_{n_k}(\omega) \notin U_{\hat{\varepsilon}}(\operatorname{cl} X^E)$. Consequently $X_{n_k}^E(\omega) \not\subset U_{\hat{\varepsilon}/2}(\operatorname{cl} X^E) \cap C$ for all $\omega \in \Omega_0$ and $n_k \geqslant n_0(\omega)$. This implies $\Gamma_{n_k}(\omega) \not\subset C$ or $(X_{n_k}^E(\omega) \setminus U_{\hat{\varepsilon}/2}(\operatorname{cl} X^E)) \cap C \neq \emptyset$, hence

$$P\left(\limsup_{n\to\infty} (A_{\hat{\varepsilon}/2}^n \cup B^n)\right) \geqslant P(\Omega_0) > 0.$$

But according to the Borel-Cantelli Lemma this inequality contradicts the assumption of the proposition. □

5. Stability of the constraint set

In this part our aim consists in deriving sufficient conditions for (VC1) and (VC2).

Let Γ be given in the following form:

$$\Gamma = \tilde{\Gamma} \cap G,$$

where

$$\tilde{\Gamma} := \{x \in \mathbb{R}^p : g^j(x) \leqslant 0, j \in \{1, \ldots, q\}\},$$

$G \subset \mathbb{R}^p$ closed, $g^j : \mathbb{R}^p \to \mathbb{R}^1$. Γ will be approximated by

$$\Gamma_n(\omega) = \tilde{\Gamma}_n(\omega) \cap G_n(\omega), \quad \tilde{\Gamma}_n(\omega) := \{x \in \mathbb{R}^p : g_n^j(x, \omega) \leqslant 0, j \in \{1, \ldots, q\}\}.$$

Here g_n^j are $(\mathscr{X}^p \otimes \mathscr{A}, \mathscr{X}^1)$-measurable functions, and G_n is a multifunction with measurable graph.

Lemma 4. $(\mathscr{X}^p \otimes \mathscr{A}, \mathscr{X}^1)$*-measurability of* g_n^j, $j \in \{1, \ldots, q\}$, *implies* $\operatorname{Graph} \tilde{\Gamma}_n \in \mathscr{A} \otimes \mathscr{X}^p$.

Proof. We have

$$\operatorname{Graph} \tilde{\Gamma}_n = \left\{(\omega, x) \in \Omega \times \mathbb{R}^p : \left\{\sup_{j\in\{1,\ldots,q\}} g_n^j(x, \omega)\right\} \cap (-\infty, 0] \neq \emptyset\right\} = \operatorname{dom} \hat{\Phi},$$

where

$$\hat{\Phi}(\omega, x) = \left\{\sup_{j\in\{1,\ldots,q\}} g_n^j(x, \omega)\right\} \cap (-\infty, 0].$$

$\hat{\Phi}$ is measurable by our assumption, hence $\operatorname{Graph} \tilde{\Gamma}_n \in \mathscr{A} \otimes \mathscr{X}^p$. □

The following theorems are stochastic versions of Theorems 3.1.2(1) and 3.1.5 in [1] with the exceptions that for convenience we confine to $\mathbb{R}^p$ and that we do not allow for a countable index set. If we have infinitely many restrictions, the assertion concerning the convergence rate cannot be achieved.

Theorem 5. *Let the following assumptions be satisfied*:

$$\text{(VgL)}\quad g_n^j \xrightarrow[\mathbb{R}^p]{\text{pl-prob}(\xi)} g^j,\quad g^j \textit{ l.s.c.},\ j\in\{1,\dots,q\};$$

$$\text{(VGU)}\quad G_n \xrightarrow{\text{u-prob}(\xi)} G.$$

Then $\Gamma_n \xrightarrow{\text{u-prob}(\xi)} \Gamma$.

Lemma 5 and Lemma 6 will be used in the proof of Theorem 5.

Lemma 5. *Let* $A\subset\mathbb{R}^p$, $B\subset R^p$ *be closed sets and suppose that* $K\in\mathcal{K}^p$. Then

$$\forall\varepsilon>0,\ \exists\delta_K>0,\quad \forall x\in K\setminus U_\varepsilon(A\cap B),$$

$$(x\in K\setminus U_{\delta_K}(A))\vee(x\in K\setminus U_{\delta_K}(B)).\qquad\square$$

Lemma 6. *Let* $K\in\mathcal{K}^p$ *and suppose that the functions* g^j, $j\in\{1,\dots,q\}$, *are l.s.c. Then*

$$\forall\delta>0,\ \exists\mu_K>0, \forall x\in C_{\delta,K},\ \exists j_0\in\{1,\dots,q\},\quad g^{j_0}(x)\geq\mu_K,$$

where $C_{\delta,K}:=\{x\in K: d(x,\tilde{\Gamma})\geq\delta\}$. $\square$

The proofs of Lemma 5 and Lemma 6 are straightforward and will be omitted.

Proof of Theorem 5. Let $\varepsilon>0$ and $K\in\mathcal{K}^p$ be given. Lemma 5 assigns to ε, K, $\tilde{\Gamma}$ and G a $\delta=\delta_K>0$. Lemma 6 guarantees the existence of a $\mu=\mu_K(\delta)>0$ such that

$$\forall x\in C_{\delta,K},\ \exists j_0\in\{1,\dots,q\},\quad g^{j_0}(x)\geq\mu.$$

We consider the sets $U\{x\}:=\operatorname{int} U^\mu\{x\}$, where $U^\mu\{x\}$ denotes the neighbourhood associated to $\{(g_n^j)_{n\in\mathbb{N}}, j\in\{1,\dots,q\}\}$, x and μ by the definition of p-lower semicontinuous in probability. $\{U\{x\}, x\in C_{\delta,K}\}$ being an open cover of $C_{\delta,K}$, we can select a finite cover $\{U\{x_l\}, l=1,\dots,k\}$.

Now let $n\in\mathbb{N}$ and ω be such that $[(\tilde{\Gamma}_n(\omega)\cap G_n(\omega))\setminus U_\varepsilon(\tilde{\Gamma}\cap G)]\cap K\neq\emptyset$. Hence there exists an $x_n(\omega)\in\tilde{\Gamma}_n(\omega)\cap G_n(\omega)$ with $x_n(\omega)\in K\setminus U_\varepsilon(\tilde{\Gamma}\cap G)$. According to the choice of δ then either $x_n(\omega)\in K\setminus U_\delta(G)$ or $x_n(\omega)\in C_{\delta,K}$. In the first case we obtain $(G_n(\omega)\setminus U_\delta(G))\cap K\neq\emptyset$ and make use of (VGU).

If $x_n(\omega)\in C_{\delta,K}$ we proceed as follows: $x_n(\omega)$ belongs to some $U\{x_l\}$, $x_l\in C_{\delta,K}$, and by Lemma 6, $g^{j_0}(x_l)\geq\mu$ for some $j_0\in\{1,\ldots,q\}$. On the other hand $g_n^j(x_n(\omega),\omega)\leq 0\ \forall j\in\{1,\ldots,q\}$. Consequently $\inf_{x\in U^{\mu}\{x_l\}} g_n^{j_0}(x,\omega)-g^{j_0}(x_l)\leq-\mu$ and finally

$$\begin{aligned}&P\{\omega\colon (\Gamma_n(\omega)\setminus U_\varepsilon(\Gamma))\cap K\neq\emptyset\}\\ &\quad\leq P\{\omega\colon (G_n(\omega)\setminus U_\delta(G))\cap K\neq\emptyset\}\\ &\qquad+\sum_{l=1}^{k}\sum_{j=1}^{q}P\left\{\omega\colon \inf_{x\in U^{\mu}\{x_l\}} g_n^j(x,\omega)-g^j(x_l)\leq-\mu\right\}=\mathrm{o}(\xi_n). \qquad\square\end{aligned}$$

Theorem 6. *Let the following assumptions be satisfied:*

(VgU) $g_n^j \xrightarrow[\tilde\Gamma^0]{\text{pu-prob}(\xi)} g^j$, g^j *u.s.c. on* $\tilde\Gamma^0$, $j\in\{1,\ldots,q\}$,

where

$$\tilde\Gamma^0:=\{x\in\mathbb{R}^p\colon g^j(x)<0\ \forall j\in\{1,\ldots,q\}\};$$

(VGL) $G_n \xrightarrow{\text{l-prob}(\xi)} G$;

(VΓ) $\Gamma\subset \operatorname{cl}\tilde\Gamma^0\cap G$.

Then $\Gamma_n \xrightarrow{\text{l-prob}(\xi)} \Gamma$.

For the proof of Theorem 6 we need some special preliminaries, too. Firstly, we introduce the sets

$$\Gamma_{\mathrm{R}}:=\{\mathrm{x}\in\mathbb{R}^{\mathrm{p}}\colon \exists \mathrm{j}_0\in\{1,\ldots,\mathrm{q}\}\text{ with } g^{j_0}(x)\geq 0\},$$

$$\mathrm{Cl}_{\delta,K}:=\{x\in K\colon d(x,\Gamma_{\mathrm{R}})\geq\delta\},\quad \delta>0,\ K\in\mathcal{K}^p.$$

Lemma 7. *Let $K\in\mathcal{K}^p$ and suppose that (VΓ) is satisfied and the functions g^j, $j\in\{1,\ldots,q\}$, are u.s.c. on $\tilde\Gamma^0$. Then*

$$\forall\varepsilon>0,\ \exists\delta_K>0,\ \forall x_0\in\Gamma\cap K,\ \exists x_1\in C\cap\tilde\Gamma^0 \textit{ with } U_{\delta_K}\{x_1\}\subset U_\varepsilon\{x_0\},$$
$$U_{\delta_K}\{x_1\}\subset\tilde\Gamma^0.$$

Proof. Assume

$$\exists\varepsilon>0,\ \forall\delta_K>0,\ \exists x_0\in\Gamma\cap K,\ \forall x_1\in G\cap\tilde\Gamma^0 \text{ with } U_{\delta_K}\{x_1\}\subset U_\varepsilon\{x_0\},$$
$$\exists x\in U_{\delta_K}\{x_1\},\quad x\notin\tilde\Gamma^0.$$

We choose a sequence $(\delta_{K,n})_{m\in\mathbb{N}}$, $\delta_{K,n}\downarrow 0$, and a corresponding sequence $(x_{0n})_{n\in\mathbb{N}}$ with $x_{0n}\in\Gamma\cap K$. W.l.o.g. we can suppose that $(x_{0n})_{n\in\mathbb{N}}$ converges: $x_{0n}\to\hat{x}_0\in\operatorname{cl}\Gamma\cap K$.

Now we consider the set $U_{\varepsilon/2}\{\hat{x}_0\}$. Let $\hat{x}_0\}$. Let $\hat{x}_1 \in U_{\varepsilon/2}\{\hat{x}_0\} \cap G \cap \tilde{\Gamma}^0$ be fixed. Then $U_{\delta_{K,n}}\{\hat{x}_1\} \subset U_\varepsilon\{x_{0n}\}$ $\forall n \geqslant n_0$. Further, according to our assumption, to $\hat{x}_1$ there exists a sequence $(x_n)_{n\in\mathbb{N}} = (x_n(\hat{x}_1))_{n\in\mathbb{N}}$ with $x_n \in U_{\delta_{K,n}}\{\hat{x}_1\}$ and $x_n \notin \tilde{\Gamma}^0$ $\forall n \geqslant n_0$. Hence $g^{j(n)}(x_n) \geqslant 0$ for some $j(n) \in \{1, \ldots, q\}$. At least one j_0 must occur infinitely often in the sequence $(j(n))_{n\in\mathbb{N}}$, therefore $g^{j_0}(x_{n_k}) \geqslant 0$, $k \in \mathbb{N}$.

Because of $x_{n_k} \to \hat{x}_1$ and the upper semicontinuity of g^j on $\tilde{\Gamma}^0$ we obtain $g^{j_0}(\hat{x}_1) \geqslant 0$. Thus for each $\hat{x}_1 \in U_{\varepsilon/2}\{\hat{x}_0\}$ either $\hat{x}_1 \notin G$ or $\hat{x}_1 \notin \tilde{\Gamma}^0$. If $\hat{x}_0 \in \Gamma$, we have a contradiction to (VΓ). In the case $\hat{x}_0 \in \mathrm{cl}\,\Gamma \setminus \Gamma$ there is an $\tilde{x}_0 \in U_{\varepsilon/4}\{\hat{x}_0\} \cap \Gamma$ such that $U_{\varepsilon/4}\{\tilde{x}_0\} \subset U_{\varepsilon/2}\{\hat{x}_0\}$ and we can conclude in the same way. □

Lemma 8. *Let $K \in \mathcal{K}^p$ and suppose that the functions g^j, $j \in \{1, \ldots, q\}$, are u.s.c. on $\tilde{\Gamma}^0$. Then*

$$\forall \delta > 0,\ \exists \mu_K > 0,\quad \forall x \in \mathrm{CI}_{\delta,K},\ \forall j \in \{1, \ldots, q\},\quad g^j(x) \leqslant -\mu_K. \qquad \square$$

The proof is straightforward and will be omitted.

Proof of Theorem 6. Let $\varepsilon > 0$ and $K \in \mathcal{K}^p$ be given. Lemma 7 assigns to ε, K and Γ a $\delta = \delta_K > 0$. Lemma 8 guarantees the existence of a $\mu = \mu_K(\delta) > 0$ such that

$$\forall x \in \mathrm{CI}_{\delta,K},\ \forall j \in \{1, \ldots, q\},\quad g^j(x) \leqslant -\mu.$$

We consider the sets $U\{x\} := U_{\delta/2}\{x\} \cap \mathrm{int}\, U^\mu\{x\}$, where $U^\mu\{x\}$ is the neighbourhood associated to $\{(g_n^j)_{n\in\mathbb{N}}, j \in \{1, \ldots, q\}\}$, x and μ by the definition of p-upper semicontinuous convergence in probability. $\{U\{x\}, x \in \mathrm{CI}_{\delta,K}\}$ being an open cover of the compact set $\mathrm{CI}_{\delta,K}$, we can select a finite cover $\{U\{x_l\}, l = 1, \ldots, k\}$. Finally, let $\delta_1 := \min_{l\in\{1,\ldots,k\}} \mathrm{rad}\, U\{x_l\}$. Now, suppose that $n \in N$ and ω is such that $(\Gamma \setminus U_\varepsilon(\Gamma_n(\omega))) \cap K \neq \emptyset$. Hence there exists an $x_0(\omega) \in \tilde{\Gamma} \cap G \cap K$ with $x_0(\omega) \notin U_\varepsilon(\tilde{\Gamma}_n(\omega) \cap G_n(\omega))$. According to Lemma 7 to $x_0(\omega)$ and K we find an $x_1(\omega) \in G \cap \tilde{\Gamma}^0 \cap U_\varepsilon(K)$ with $U_\delta\{x_1(\omega)\} \subset U_\varepsilon\{x_0(\omega)\}$ and $U_\delta\{x_1(\omega)\} \subset \tilde{\Gamma}^0$. $x_1(\omega)$ belongs to some $U\{x_l\}$, $x_l \in \mathrm{CI}_{\delta,K}$. Making use of Lemma 8 we obtain $g^j(x_l) \leqslant -\mu$ $\forall j \in \{1, \ldots, q\}$. Assume that $U\{x_l\} \cap G_n(\omega) = \emptyset$. Hence $(G \setminus U_{\delta_1}(G_n(\omega))) \cap \mathrm{cl}\, U_\varepsilon(K) \neq \emptyset$. Otherwise there is an $x_n(\omega) \in U\{x_l\} \cap G_n(\omega)$. Since $U\{x_l\} \subset U_\delta\{x_1(\omega)\} \subset U_\varepsilon\{x_0(\omega)\}$, $x_n(\omega)$ cannot belong to $\tilde{\Gamma}_n(\omega)$. Consequently there exists a $j_0 \in \{1, \ldots, q\}$ such that $g_n^{j_0}(x_n(\omega), \omega) > 0$ and finally

$$\sup_{x \in U^\mu\{x_l\}} g_n^{j_0}(x, \omega) - g^{j_0}(x_l) > \mu.$$

Summarizing,

$$\begin{aligned} &P\{\omega\colon (\Gamma \setminus U_\varepsilon(\Gamma_n(\omega))) \cap K \neq \emptyset\} \\ &\quad \leqslant P\{\omega\colon (G \setminus U_{\delta_1}(G_n(\omega))) \cap \mathrm{cl}\, U_\varepsilon(K) \neq \emptyset\} \\ &\qquad + \sum_{l=1}^{k} \sum_{j=1}^{q} P\left\{\omega\colon \sup_{x \in U^\mu\{x_l\}} g_n^j(x, \omega) - g^j(x_l) \geqslant \mu\right\} = \mathrm{o}(\xi_n). \qquad \square \end{aligned}$$

6. Stochastic programming problems with chance constraints

Problems where the expectation of a random function is to be minimized with respect to probabilistic constraints play an important role in stochastic programming. Therefore we shall investigate them in more detail.

Suppose that we are given the following problem:

$$\text{(P)} \qquad \min_{x\in\Gamma} E\boldsymbol{\phi}(x, Z),$$

where

$$\Gamma=\tilde{\Gamma}=\{x\in\mathbb{R}^p\colon \eta^j\leq P\{\omega\colon \gamma_l^j(x, Z(\omega))\leq 0,\ l\in\{1,\ldots,q_j\}\},\ j\in\{1,\ldots,q\}\}.$$

$\boldsymbol{\phi}$ may be vector-valued: $\boldsymbol{\phi}(x,z)=(\phi^1(x,z),\ldots,\phi^r(x,z))^{\mathrm{T}}$. The functions ϕ^j, $\gamma_l^j\colon\mathbb{R}^p\times\mathbb{R}^m\to\mathbb{R}^1$ are supposed to be measurable with respect to the second variable. $Z\colon[\Omega,\mathscr{A},P]\to[\mathbb{R}^m,\mathscr{X}^m]$ is a random variable; $\eta^j\in(0,1)$. E denotes the expectation with respect to the probability measure P_Z on $[\mathbb{R}^m,\mathscr{X}^m]$, which is induced by P. $E\boldsymbol{\phi}(x,Z)$ is assumed to exist for all $x\in\Gamma$.

Often the distribution function F_Z of Z is not completely known and one has to deal with estimations for the whole distribution function or at least certain parameters of it. Therefore we shall consider the cases that

(i) the distribution function of Z is estimated by the empirical distribution function and

(ii) certain parameters of the distribution function have to be estimated.

However, before dealing with the estimators, we shall investigate what conditions concerning ϕ^j, γ_l^j and F_Z are sufficient for the assumptions on the original problem.

6.1. Conditions concerning the original problem

We have the following correspondences:

$$f^j(x)=E\phi^j(x,Z)$$

and

$$g^j(x)=\eta^j-P\{\omega\colon\gamma_l^j(x,Z(\omega))\leq 0,\ l\in\{1,\ldots,q_j\}\}=E(\eta^j-\chi_{M^j(x)}(Z)),$$

where

$$M^j(x):=\{z\in\mathbb{R}^m\colon\gamma_l^j(x,z)\leq 0,\ l\in\{1,\ldots,q_j\}\}$$

and

$$\chi_\Lambda(z):=\begin{cases}1 & \text{if } z\in\Lambda,\\ 0 & \text{otherwise.}\end{cases}$$

We start by investigating the continuity properties of f^j.

Proposition 5. *Let $\phi^j(\cdot, z)$ be l.s.c. at x_0 for P_Z-almost all $z \in \mathbb{R}^m$ and suppose that one of the following conditions is satisfied for $j \in \{1, \ldots, q\}$:*

(L1) *There exists a neighbourhood $U\{x_0\}$ such that*

$$E \inf_{x \in U\{x_0\}} \phi^j(x, Z) > -\infty.$$

(L2) *There exists a neighbourhood $U\{x_0\}$ such that $\phi^j(\cdot, z)$ is convex on $U\{x_0\}$ for P_Z-almost all z.*

Then f^j is l.s.c. at x_0.

Moreover, if $\phi^j(\cdot, z)$ is l.s.c. at x_0 for P_Z-almost all $z \in \mathbb{R}^m$ and (L1) *is fulfilled, the equation*

$$\sup_{U \in \mathcal{U}\{x_0\}} E \inf_{x \in U} \phi^j(x, Z) = E\phi^j(x_0, Z)$$

is valid, where $\mathcal{U}\{x_0\}$ denotes the family of open neighbourhoods U of x_0 with $U \subset U\{x_0\}$. □

The first assertion with the supplement follows from Lemma 3 in [18]. The considerations there remain true if we have lower semicontinuity at x_0 for P_Z-almost all (and not all) z only. Compare also [38]. The second assertion can be derived from Proposition 2.2 in [38].

The conditions (VE1) and (VE2) require convexity of f^j. It is well-known that convexity or strict convexity (as a sufficient condition for strict explicit quasiconvexity) of $\phi^j(\cdot, z)$ entail the corresponding properties for $E\phi^j(\cdot, Z)$. However, if the functions $\phi^j(\cdot, z)$ are strictly explicitly quasiconvex, this property cannot be expected for $E\phi^j(\cdot, Z)$, consider for example $\phi^j(x, z) = zx$, $x \in \mathbb{R}^1$, $P_Z\{1\} = P_Z\{-1\} = \frac{1}{2}$.

Γ is to be closed and sometimes convex or strictly convex. Since the lower semicontinuity of g^j, $j \in \{1, \ldots, q\}$, implies the closedness of Γ we shall continue with investigating semicontinuity properties of g^j.

It is convenient to employ Proposition 5. (L1) being satisfied for the functions $\chi_{M^j(\cdot)}(\cdot)$, we need assumptions assuring that these functions are semicontinuous P_Z-almost surely at a given point x_0.

Proposition 6. *If the functions $\gamma_l^j(\cdot, z)$, $j \in \{1, \ldots, q\}$, $l \in \{1, \ldots, q_j\}$, are l.s.c. at x_0 for all $z \in \mathbb{R}^m$, then the functions $\chi_{M^j(\cdot)}(z)$ are u.s.c. at x_0 for all $z \in \mathbb{R}^m$ and the functions g^j are l.s.c. at x_0.*

Proof. Let $z \in \mathbb{R}^m$ be given. If $\chi_{M^j(x_0)}(z) = 1$ we have nothing to show. Now, assume that $\chi_{M^j(x_0)}(z) = 0$. Hence there is an $l_0 \in \{1, \ldots, q_j\}$ with $\gamma_{l_0}^j(x_0, z) > 0$. $\gamma_{l_0}^j(\cdot, z)$ being l.s.c. at x_0, there is a neighbourhood $U\{x_0\}$ with $\gamma_{l_0}^j(x, z) > 0$ $\forall x \in U\{x_0\}$. Consequently $\chi_{M^j(x)}(z) = 0$ $\forall x \in U\{x_0\}$. □

Proposition 7. *Let the following conditions be fulfilled:*

(L3) *The functions* $\gamma_l^j(\cdot, z)$ *are u.s.c. at* x_0 *for all* $z \in \mathbb{R}^m$, *the functions* $\gamma_l^j(x_0, \cdot)$ *are u.s.c. on* $\mathbb{R}^m$, $j \in \{1, \ldots, q\}$, $l \in \{1, \ldots, q_j\}$;

(L4) $M^j(x_0) \subset \mathrm{cl}\{z \in \mathbb{R}^m \colon \gamma_l^j(x_0, z) < 0 \ \forall l \in \{1, \ldots, q_j\}\}$, $j \in \{1, \ldots, q\}$;

(L5) F_Z *is absolutely continuous.*

Then the functions $\chi_{M^j(\cdot)}(z)$ *are l.s.c. at* x_0 *for* P_Z*-almost all* z *and the functions* g^j *are u.s.c. at* x_0.

Proof. Let $j \in \{1, \ldots, q\}$ be fixed and

$$\zeta_0^j := \{z \in \mathbb{R}^m \colon \exists l \in \{1, \ldots, q_j\} \text{ with } \gamma_l^j(x_0, z) = 0\} \cap M^j(x_0).$$

We have $\operatorname{int} M^j(x_0) = \{z \in \mathbb{R}^m \colon \gamma_l^j(x_0, z) < 0, \ l \in \{1, \ldots, q_j\}\}$. Hence $\mathscr{L}\{z \in \mathbb{R}^m \colon \gamma_l^j(x_0, z) \leq 0, \ l \in \{1, \ldots, q_j\}\} = \mathscr{L}\{z \in \mathbb{R}^m \colon \gamma_l^j(x_0, z) < 0, \ l \in \{1, \ldots, q_j\}\}$, where $\mathscr{L}$ denotes the Lebesgue-measure on $[\mathbb{R}^m, \mathscr{X}^m]$. P_Z being absolutely continuous with respect to $\mathscr{L}$, we obtain $P_Z(\zeta_0^j) = 0$.

Now we investigate the lower semicontinuity of $\chi_{M^j(\cdot)}(z)$ at x_0. Let $z \in \mathbb{R}^m$ be given. If $\chi_{M^j(x_0)}(z) = 0$ there is nothing to show. Otherwise we distinguish the case $z \in \zeta_0^j$ and $z \notin \zeta_0^j$. For $z \in \zeta_0^j$ in general we cannot expect that $\chi_{M^j(\cdot)}(z)$ is l.s.c. at x_0. If $z \notin \zeta_0^j$, we have $\gamma_l^j(x_0, z) < 0 \ \forall l \in \{1, \ldots, q_j\}$, hence $\gamma_l^j(\tilde{x}, z) < 0$ for all $\tilde{x}$ belonging to a neighbourhood $U\{x_0\}$ and all $l \in \{1, \ldots, q_j\}$. Thus $\chi_{M^j(\tilde{x})}(z) = 1 \ \forall \tilde{x} \in U\{x_0\}$ and the assertion follows. □

Convexity of Γ has been investigated in several papers [12, 22, 38]. In this framework quasiconcave and logarithmic concave probability measures play a crucial role. A probability measure P_Z on $[\mathbb{R}^m, \mathscr{X}^m]$ is said to be quasiconcave (logarithmic concave) if for any pair V_1, V_2 of convex subsets of $\mathbb{R}^m$ and any $\lambda \in (0, 1)$ the relation

$$P_Z(\lambda V_1 + (1-\lambda) V_2) \geq \min\{P_Z(V_1), P_Z(V_2)\}$$

$$(P_Z(\lambda V_1 + (1-\lambda) V_2) \geq [P_Z(V_1)]^{\lambda} \cdot [P_Z(V_2)]^{1-\lambda})$$

holds.

Proposition 8 is proved in [38].

Proposition 8. *Let* γ_l^j, $j \in \{1, \ldots, q\}$, $l \in \{1, \ldots, q_j\}$, *be convex and* P_Z *be quasiconcave. Then* Γ *is a closed convex set.* □

The conclusion of Proposition 8 still holds if the convexity of the functions γ_l^j is replaced by quasiconvexity and continuity.

The following statement is an auxiliary result which will be used to derive further properties of Γ.

Lemma 9. *Let P_Z be quasiconcave and such that $P_Z(V)>0$ for all Borel sets V with* int $V\neq\emptyset$. *Moreover, assume that the functions γ_l^j, $j\in\{1,\ldots,q\}$, $l\in\{1,\ldots,q_j\}$, are continuous and have one of the following properties*:

(L6) *γ_l^j strictly explicitly quasiconvex*;

(L7) *$\gamma_l^j(x,z)=\gamma_1^{j,l}(x)+\gamma_2^{j,l}(z)$, where $\gamma_1^{j,l}$ is strictly explicitly quasiconvex and $\gamma_2^{j,l}$ is quasiconvex.*

Then for each $x_\lambda=\lambda x_0+(1-\lambda)x_1$, $x_0\in\Gamma$, $x_1\in\Gamma$, $x_0\neq x_1$, $\lambda\in(0,1)$, the inequality $g^j(x_\lambda)<0$ $\forall j\in\{1,\ldots,q\}$ holds.

Proof. Let $x_0\in\Gamma$, $x_1\in\Gamma$, $x_0\neq x_1$, $\lambda\in(0,1)$ and $j\in\{1,\ldots,q\}$. Choose $z_0\in M^j(x_0)$ and $z_1\in M^j(x_1)$ and consider $z_\lambda:=\lambda z_0+(1-\lambda)z_1$. In both cases we obtain

$$\gamma_l^j(x_\lambda,z_\lambda)<\max\{\gamma_l^j(x_0,z_0),\gamma_l^j(x_1,z_1)\}\quad\forall l\in\{1,\ldots,q_j\}.$$

Hence $z_\lambda\in M^j(x_\lambda)$ and, furthermore, because of the continuity of γ_l^j to each z_λ there is a neighbourhood $U\{z_\lambda\}$ such that $U\{z_\lambda\}\subset M^j(x_\lambda)$. Therefore either $P_Z(M^j(x_\lambda))=1$ or

$$\begin{aligned}P_Z(M^j(x_\lambda))&>P_Z(\lambda M^j(x_0)+(1-\lambda)M^j(x_1))\\&\geq\min\{P_Z(M^j(x_0)),P_Z(M^j(x_1))\}\geq\eta^j,\end{aligned}$$

and consequently $g^j(x_\lambda)=\eta^j-P_Z(M^j(x_\lambda))<0$. □

Proposition 9. *Suppose that the assumptions of Lemma 9 are fulfilled and that the functions g^j, $j\in\{1,\ldots,q\}$, are u.s.c. Then Γ is strictly convex.*

Proof. Let $x_0\in\Gamma$, $x_1\in\Gamma$, $x_0\neq x_1$, $\lambda\in(0,1)$ be given. Making use of Lemma 9 and the semicontinuity of g^j, to $x_\lambda=\lambda x_0+(1-\lambda)x_1$ we find a neighbourhood $U\{x_\lambda\}$ such that $g^j(x)<0$ $\forall j\in\{1,\ldots,q\}$, $\forall x\in U\{x_\lambda\}$, hence $U\{x_\lambda\}\subset\Gamma$. □

Finally we shall deal with condition (VΓ). Proposition 10 is due to Wang [36].

Proposition 10. *Let the following assumptions be satisfied*:

(L8) *The functions γ_l^j, $j\in\{1,\ldots,q\}$, $l\in\{1,\ldots,q_j\}$, are convex*;

(L9) *P_z is logarithmic concave*;

(L10) *there exists an $x_0\in\Gamma$ with $g^j(x_0)<0$ $\forall j\in\{1,\ldots,q\}$.*

Then (VΓ) *is fulfilled.* □

It is easy to see that the assumptions of Lemma 9 are sufficient for (VΓ), too, if Γ contains more than one element.

If we cannot assume that P_Z is logarithmic concave or quasiconcave, we still have Proposition 11.

Proposition 11. *Let* (L5) *and the following assumptions be fulfilled*:

(L11) *P_Z has convex support*;

(L12) *the functions $\gamma_l^j(\cdot,z)$ are strictly quasiconvex and continuous for P_Z-almost all z, the functions $\gamma_l^j(x,\cdot)$ are u.s.c. $\forall x\in\Gamma$, $j\in\{1,\ldots,q\}$, $l\in\{1,\ldots,q_j\}$*;

(L13) *to each* $x \in \Gamma$ *there exists a* $d_x \in \mathbb{R}^p$ *with* $\gamma_l^j(x+d_x, z) < \gamma_l^j(x, z)$ *for* P_Z*-almost all* z *and all* $j \in \{1, \ldots, q\}$, $l \in \{1, \ldots, q_j\}$.

Then ($\mathrm{V}\Gamma$) *holds.*

Proof. Fix $x \in \Gamma$ and consider the sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n = x + \lambda_n d_x$, $\lambda_n \downarrow 0$. Because of (L12) and (L13) we have $\gamma_l^j(x_n, z) < 0$ $\forall z \in M^j(x)$, $j \in \{1, \ldots, q\}$, hence $x_n \in \Gamma$ $\forall n \geq n_0$. If $P_Z(M^j(x)) = 1$ the inequality $g^j(x_n) < 0$ immediately follows. Otherwise, let $M^j(x) \cap \zeta \subsetneqq \zeta$, where ζ denotes the support of P_Z. Then there exists a $z_0 \in \operatorname{bd} M^j(x) \cap \operatorname{r.int} \zeta$. Due to (L12) we find neighbourhoods $U_n\{z_0\}$ such that $\gamma_l^j(x_n, z) < 0$ $\forall z \in M^j(x) \cup U_n\{z_0\}$. $M^j(x)$ being closed and convex, $P_Z(U_n\{z_0\} \setminus M^j(x)) > 0$, hence

$$P_Z\{z \in \mathbb{R}^m : \gamma_l^j(x_n, z) \leq 0\ \forall l \in \{1, \ldots, q_j\}\} > \eta^j. \qquad \square$$

Note that (L13) is for instance satisfied if there exists an $x_0 \in \mathbb{R}^p$ with $\gamma_l^j(x_0, z) < 0$ for P_Z-almost all z and all $j \in \{1, \ldots, q\}$, $l \in \{1, \ldots, q_j\}$.

6.2. Assumptions concerning the estimations

A general assumption throughout the paper is the $(\mathscr{X}^p \otimes \mathscr{A}, \mathscr{X}^r)$-measurability of $\boldsymbol{f}_n$ (and $\boldsymbol{g}_n$). This condition is for instance satisfied if the functions f_n^j, $j \in \{1, \ldots, r\}$, are normal integrands, i.e. the $f_n^j(\cdot, \omega)$ are l.s.c. for all $\omega \in \Omega$ and the epigraphical multifunctions are measurable (cf. [24]). In particular, functions f_n^j which are continuous in x and measurable with respect to the second variable are normal integrands.

If the functions f_n^j are almost deterministic, i.e.

$$f_n^j(\cdot, \omega) = \tilde{f}_n^j(\cdot) \quad P\text{-a.e.}, \qquad \text{where } \tilde{f}_n^j : \mathbb{R}^p \to \mathbb{R}^1,$$

the occurring events are measurable because of the completeness of $[\Omega, \mathscr{A}, P]$. Functions f_n^j and f^j with the property

$$f_n^j(x, \omega) \equiv \hat{f}^j(x, \lambda_n),$$

$$f^j(x, \omega) \equiv \hat{f}^j(x, \lambda_0), \quad \lambda_n \to \lambda_0, \quad \hat{f}^j : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^1,$$

enable us to compare our results with those from multiobjective parametric programming:

It is easy to see that

$$(\hat{f}^j \text{ l.s.c. on } M \times \{\lambda_0\}) \Rightarrow (f_n^j \xrightarrow[M]{\text{pl-prob}(\xi)} f^j),$$

$$(\hat{f}^j \text{ u.s.c. on } M \times \{\lambda_0\}) \Rightarrow (f_n^j \xrightarrow[M]{\text{pu-prob}(\xi)} f^j),$$

with an arbitrary convergence rate ξ.

Therefore we may derive convergence assertions in the "almost surely" sense, compare the remark at the end of Section 4.

In the following we shall discuss the announced cases (i) and (ii).

(i) We assume that the distribution function of Z is estimated by the empirical distribution function. Then we obtain the estimates

$$f_n^j(x,\omega)=\frac{1}{n}\sum_{i=1}^{n}\phi^j(x,Z_i(\omega)),\quad j\in\{1,\dots,r\},$$

$$g_n^j(x,\omega)=\frac{1}{n}\sum_{i=1}^{n}[\eta^j-\chi_{M^j(x)}(Z_i(\omega))],\quad j\in\{1,\dots,q\},$$

where the Z_i are i.i.d. random variables.

If ϕ^j is $(\mathscr{L}^p\otimes\mathscr{L}^m,\mathscr{L}^1)$-measurable (for instance a normal integrand), then the functions f_n^j, $n\in\mathbb{N}$, are $(\mathscr{L}^p\otimes\mathscr{A},\mathscr{L}^1)$-measurable. Because of

$$\{(x,\omega)\in\mathbb{R}^p\times\Omega\colon\chi_{M^j(x)}(Z(\omega))=1\}=\left\{(x,\omega)\colon\sup_{l\in\{1,\dots,q_j\}}\gamma_l^j(x,Z(\omega))\leq 0\right\},$$

the $(\mathscr{L}^p\otimes\mathscr{L}^m,\mathscr{L}^1)$-measurability of γ_l^j, $l\in\{1,\dots,q_j\}$, implies the desired measurability of $\chi_{M^j(\cdot)}(\cdot)$. Thus we are ready to give sufficient conditions for $f_n^j\xrightarrow[M]{\text{pl-prob}(\xi)}f^j$ ($g_n^j\xrightarrow[M]{\text{pl-prob}(\xi)}g^j$).

Proposition 12. *Let the following assumptions be satisfied for some $s>1$:*

(L14) *$\phi^j(\cdot,z)$ is l.s.c. at x_0 for P_Z-almost all z;*

(L15) *there exists a neighbourhood $U\{x_0\}$ such that*

$$E\left|\inf_{x\in U\{x_0\}}\phi^j(x,Z)\right|^s<\infty.$$

Then

$$\forall\varepsilon>0,\ \exists U^\varepsilon\{x_0\}\in\mathscr{K}^p,$$

$$P\left\{\omega\colon\inf_{x\in U^\varepsilon\{x_0\}}f_n^j(x,\omega)-f^j(x_0)\leq-\varepsilon\right\}=o(n^{-s+1}).$$

Proof. Let $\varepsilon>0$ be given. Due to Proposition 5, (L1), we find a neighbourhood $U^\varepsilon\{x_0\}\subset U\{x_0\}$ with

$$E\inf_{x\in U^\varepsilon\{x_0\}}\phi^j(x,Z)\geq f^j(x_0)-\tfrac{1}{2}\varepsilon.$$

Now, let ω be such that $\inf_{x\in U^\varepsilon\{x_0\}}f_n^j(x,\omega)-f^j(x_0)\leq-\varepsilon$, hence profiting by the special for of f_n^j,

$$\frac{1}{n}\sum_{i=1}^{n}\inf_{x\in U^\varepsilon\{x_0\}}\phi^j(x,Z_i(\omega))-\inf_{x\in U^\varepsilon\{x_0\}}\phi^j(x,Z)$$

$$\leq\inf_{x\in U^\varepsilon\{x_0\}}\frac{1}{n}\sum_{i=1}^{n}\phi^j(x,Z_i(\omega))-f^j(x_0)\leq-\tfrac{1}{2}\varepsilon.$$

Thus

$$P\left\{\omega: \inf_{x\in U^{\varepsilon}\{x_0\}} f_n^j(x,\omega)-f^j(x_0)\leqslant -\varepsilon\right\}$$

$$\leqslant P\left\{\omega: \frac{1}{n}\sum_{i=1}^{n} \inf_{x\in U^{\varepsilon}\{x_0\}} \phi^j(x, Z_i(\omega)) - E \inf_{x\in U^{\varepsilon}\{x_0\}} \phi^j(x,Z)\leqslant -\tfrac{1}{2}\varepsilon\right\}$$

and it remains to apply Lemma 2 in [3]. □

It is clear from the proof that using other large deviations results instead of Lemma 2 in [3] we can derive further assertions even for dependent random variables Z_i.

(ii) We assume that the functions under consideration depend on an unknown parameter $y_0\in\mathbb{R}^u$ which is estimated by a sequence $(Y_n)_{n\in\mathbb{N}}$ of estimation functions $Y_n:[\Omega,\mathscr{A},P]\to[\mathbb{R}^u,\mathscr{X}^u]$. Then

$$f^j(x)=\tilde{f}^j(x,y_0),\quad f_n^j(x,\omega)=\tilde{f}^j(x,Y_n(\omega)),\qquad j\in\{1,\dots,r\},$$
$$g^j(x)=\tilde{g}^j(x,y_0),\quad g_n^j(x,\omega)=\tilde{g}^j(x,Y_n(\omega)),\qquad j\in\{1,\dots,q\},$$

where $\tilde{f}^j, \tilde{g}^j:[\mathbb{R}^p\times\mathbb{R}^u, \mathscr{X}^p\otimes\mathscr{X}^u]\to[\mathbb{R}^1,\mathscr{X}^1]$.

In our setting of probabilistic constrained stochastic programs f^j and g^j may originate from

$$f^j(x)=\int_{\mathbb{R}^m}\phi^j(x,z)\,\mathrm{d}P_Z(y_0,z),$$

$$g^j(x)=\int_{\mathbb{R}^m}\eta^j-\chi_{M^j(x)}(z)\,\mathrm{d}P_Z(y_0,z),$$

and f_n^j, g_n^j from

$$f_n^j(x,\omega)=\int_{\mathbb{R}^m}\phi^j(x,z)\,\mathrm{d}P_Z(Y_n(\omega),z),$$

$$g_n^j(x,\omega)=\int_{\mathbb{R}^m}\eta^j-\chi_{M^j(x)}(z)\,\mathrm{d}P_Z(Y_n(\omega),z),$$

where $P_Z(Y_n,\cdot)$ denotes the conditional distribution with respect to Y_n of the random variable Z. Consequently f_n^j and g_n^j are measurable with respect to the second variable.

If $\phi^j(\cdot,z)$ and $\chi_{M^j(\cdot)}(z)$ are continuous in each $x\in\mathbb{R}^p$ for all z and assumption (L1) is satisfied for Φ^j and $-\Phi^j$, *then* $f_n^j(\cdot,\omega)$ and $g_n^j(\cdot,\omega)$ are continuous, hence f_n^j and g_n^j are normal integrands.

Proposition 13. *Let the following assumptions be satisfied*:

(L16) *The functions* $\tilde{f}^j$ *are l.s.c. at* (x_0,y_0);

(L17) *there exists a convergence rate* ξ *such that*

$$\forall\delta>0,\quad P\{\omega: \|Y_n(\omega)-y_0\|\geqslant\delta\}=\mathrm{o}(\xi_n).$$

Then

$$\forall\varepsilon>0,\ \exists U^{\varepsilon}\{x_0\}\in\mathscr{K}^p,\quad P\left\{\omega: \inf_{x\in U^{\varepsilon}\{x_0\}} f_n^j(x,\omega)-f^j(x_0)\geqslant-\varepsilon\right\}=\mathrm{o}(\xi_n).$$

Proof. Let $\varepsilon > 0$ be given. $\tilde{f}^j$ being l.s.c. at (x_0, y_0), to $\frac{1}{2}\varepsilon$ there is a $\delta > 0$ with

$$\tilde{f}^j(x, y) - \tilde{f}^j(x_0, y_0) \geq -\tfrac{1}{2}\varepsilon \quad \forall (x, y) \in \mathrm{cl}\, U_\delta\{x_0\} \times \mathrm{cl}\, U_\delta\{y_0\}.$$

Now, let ω be such that for $U^\varepsilon\{x_0\} := \mathrm{cl}\, U_\delta\{x_0\}$,

$$\inf_{x \in U^\varepsilon\{x_0\}} \tilde{f}^j(x, Y_n(\omega)) - \tilde{f}^j(x_0, y_0) \leq -\varepsilon.$$

Then $\| Y_n(\omega) - y_0 \| \geq \delta$.

Consequently

$$P\left\{ \omega\colon \inf_{x \in U^\varepsilon\{x_0\}} f_n^j(x, \omega) - f^j(x_0) \leq -\varepsilon \right\}$$
$$\leq P\{\omega\colon \| Y_n(\omega) - y_0 \| \geq \delta\} = \mathrm{o}(\xi_n). \qquad \square$$

Sufficient conditions for (L17) may be found in the literature on large deviations in statistics, for instance in [3, 18] and many other papers.

Acknowledgement

The author would like to thank an anonymous referee for helpful advices.

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